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ABSTRACT

The fourth volume of the 24th annual conference of the International Group for the Psychology of Mathematics Education contains full research report papers. Papers include: (1) "What are essential to apply the 'discovery' function of proof in lower secondary school mathematics?" (Mikio Miyazaki); (2) "The anatomy of an 'open' mathematics lesson" (Ida Ah Chee Mok); (3) "Interview-based assessment of early multiplication and division" (Joanne Mulligan and Robert [Bob] Wright); (4) "Procedures of finding a solution for word problems: A study of Mozambican secondary school students" (Adelino Evaristo Murimo); (5) "Proto-mathematical concepts in Northern Chilean Aymaras" (Vicente Neumann, Rafael Nunez, and Manuel Mamani); (6) "The case analysis of six graders' journal writings: Using the 'Framework for Analyzing the Quality of Transactional Writing'" (Hiroyuki Ninomiya); (7) "Listen to the graph: Children's matching of melodies with their visual representations" (Steven Nisbet and John Bain); (8) "Teaching mathematics via internet: Written interactions between tutor and student" (Lourdes Figueiras Ocana); (9) "Elementary school students' statistical thinking: An international perspective" (Bob Perry, Ian J. Putt, Graham A. Jones, Carol A. Thornton, Cynthia W. Langrall, and Edward S. Mooney); (10) "The properties of necessity and sufficiency in the construction of geometric figures with Cabri" (Angela Pesci); (11) "Students' processes of symbolizing in algebra: A semiotic analysis of the production of signs in generalizing tasks" (Luis Radford); (12) "Student perceptions of variation in a sampling situation" (Chris Reading and Mike Shaughnessy); (13) "Capstone courses in problem solving for prospective secondary teachers: Effects on beliefs and teaching practices" (Cheryl Roddick, Joanne Rossi Becker, and Barbara J. Pence); (14) "Experiencing the necessity of a mathematical statement" (Catherine Sackur, Jean-Philippe Drouhard, Maryse Maurel, and Teresa Assude); (15) "Indexicality and reflexivity in the documentary of classroom construction of ratio concept" (Tetsuro Sasaki); (16) "The effect of mapping analogical subtraction procedures on conceptual and procedural knowledge" (Bracha Segalis and Irit Peled); (17) "A teaching experiment on mathematical proof: Roles of metaphor and externalization" (Yasuhiro Sekiguchi); (18) "Metacognition: The role of the 'inner teacher' (6): Research on the relation between a transfiguration of student's mathematics knowledge and 'inner teacher'" (Keiichi Shigematsu

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and Yoshio Katsumi); (19) "An analysis of 'make an organized list' strategy in problem solving process" (Norihiro Shimizu); (20) "Explaining your solution to younger children in a written assessment task" (Yoshinori Shimizu); (21) "Reconsidering mathematical validation in the classroom" (Martin A. Simon); (22) "The forced autonomy of mathematics teachers" (Jeppe Skott); (23) "The genesis of new mathematical knowledge as a social construction" (Heinz Steinbring); (24) "Modalities of students' internal frames of reference in learning school mathematics" (Hitoshi Takahashi); (25) "The effects of metacognitive training in mathematical word problem solving in a computer environment" (Su-Kwang Teong, John Threlfall, and John Monaghan); (26) "Using algebraic processes to promote concept development" (Anne R. Teppo); (27) "Children's learning of independence: Can research help?" (John M. Truran); (28) "Advancing arithmetic thinking based on children's cultural conceptual activities: The Pick-Red-Point game" (Wen Huan Tsai); (29) "The intuitive rule same A--same B: The case of area and perimeter" (Pessia Tsamir and Nurit Mandel); (30) "Intuitive beliefs and undefined operations: The cases of division by zero" (Pessia Tsamir and Dina Tirosh); (31) "Factors contributing to learning of calculus" (Behiye Ubuz and Burcu Kirkpinar); (32) "Supporting change through a mathematics team forum for teachers' professional development" (Paola Valero and Kristine Jess); (33) "Student teachers' concept images of algebraic expressions" (Nelis Vermeulen); (34) "Spreadsheet mathematics in college and in the workplace: A mediating instrument?" (Geoff D. Wake, Julian S. Williams, and J. Haighton); (35) "Visualisation and the development of early understanding in algebra" (Elizabeth Warren); (36) "Linguistic aspects of computer algebra systems in higher mathematics education" (Carl Winslow); (37) "Two patterns of progress of problem-solving process: From a representational perspective" (Atsushi Yamada); and (38) "Student optimism, pessimism, motivation, and achievement in mathematics: A longitudinal study" (Shirley M. Yates). (ASK)

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of the

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**Editors: Tadao Nakahara
and Masataka Koyama
Hiroshima University**

Volume 4

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Editors:
Tadao Nakahara
Masataka Koyama

Department of Mathematics Education
Hiroshima University
1-1-2, Kagamiyama, Higashi-Hiroshima 739-8523
Japan

Fax: +81-824-22-7076
E-mail: nakahar@hiroshima-u.ac.jp
E-mail: mkoyama@hiroshima-u.ac.jp

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RESEARCH REPORTS

WHAT ARE ESSENTIAL TO APPLY THE "DISCOVERY" FUNCTION OF PROOF IN LOWER SECONDARY SCHOOL MATHEMATICS?

MIKIO MIYAZAKI[*]
Department of Mathematics Education, Faculty of Education, Shinshu University, Japan

ABSTRACT. This study found several conditions to apply the "discovery" function of proof in lower secondary school mathematics. These conditions are classified into three kinds of quality: an initial problem and its proof (5.1), an additional problem (5.2), and individual activities to solve it (5.3). In order to conclude them, Chapter 3 described what the "discovery" function of proof was (3.1), and why we should have focus on it (3.2). Chapter 4 showed experimental methods in cooperation with 8th graders. Chapter 5 described why each prerequisite was essential in order to apply the "discovery" function of proof with reference to 8th graders' individual activities.

1. Focusing on making new results with mathematical proof

The importance of proof in school mathematics has an inclination to increase on account of sharp changes of society (e.g. information-oriented, globalizations) and inevitable national-wide reforms of education as responses. In order to enhance the inclination toward a desirable direction, proof researches in mathematics education are responsible to reconsider the importance of teaching and learning a proof, and to contribute the substantial enrichments. The enrichments of teaching and learning a proof comprise more conscious applications of the functions of proof, which will be helpful for students to make their life more rational by nature. Hence, mathematics educators should make an effort to infiltrate the functions of proof into the whole school mathematics curriculum.

This study will focus on making new results with mathematical proof among the functions of proof, which de Villeres (1990) called "discovery". The reason is as follows. In Japan, the upper graders of elementary school are intended to develop the naive foundations of mathematical proof through saying the reason of his/her ideas where the qualities of reasoning and representation are not concerned. 8th graders are firstly intended to learn a meaning of mathematical proof, the structural relations among substantial elements (hypothesis, conclusion, assumption, etc.), and how to make a mathematical proof in plane geometry. They are also intended to appreciate a mathematical proof as a useful tool to investigate geometrical figures. On the other hand, students have less opportunity to make new results with mathematical proof, although they usually have more experiences to justify a proposition, persuade the others, and make local organization of propositions slightly. However, making new results with mathematical proof can help students to make their activities more productive and creative even in the situations outside school mathematics.

2. Aims and methods

This study will solve the following problem.

What are essential to apply the "discovery" function of proof in lower secondary school mathematics?

Solving this problem has the following importance. The "discovery" function of proof had taken an important role in mathematics, and it could also in mathematics education (de Villiers, 1990). Then, previous researches focused on how to apply the function into school mathematics. For instance, de Villers (1998) described some high school students' activities with "Sketchpad" in which they challenged geometrical

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problems concerning the following issues: a quadrilateral constructed with the midpoints of sides of a kite, a center of gravity of a triangle, Fermat's point of a right triangle, and interior and exterior angles of quadrilaterals. With each worksheet, students were led to a series of activities as follows: Constructing figures with "Sketchpad" → Conjecturing propositions → Verifying propositions with "Sketchpad" → Improving propositions → Showing why a proposition is true → Comparing his/her proof with others. He pointed out the expected students' activities with the "discovery" function, and reported that they reconsidered the definitions of quadrilaterals and interior angles through coping with concave or 'crossed' quadrilaterals. Balacheff (1991) described how pairs of students treated refutations of their propositions and proofs about the number of diagonals of a polygon, reported that some pairs changed their definition of polygon and/or diagonals on the basis of their proof, and specified three factors which dominated students' treatment of refutation as follows: the analysis with reference to the problem itself, the analysis with reference to a global conception of what mathematics consists of, the analysis with reference to the situation (p.107).

These previous researches show the fact that the "discovery" function of proof can be applicable to school mathematics, and that the application will make it possible for students to access the nature of proving activities in mathematics. On the other hand, they were limited to analyze each case of students' activities or to find out some critical factors concerning the "discovery" function. The prerequisites to apply the function successfully are not identified.(In order to solve the previous problem, this study will develop its discussion as described in "ABSTRACT".)

3. Preparation

3.1 What is the "discovery" function of proof?

As the functions of proof in mathematics, de Villiers (1990) identified five functions as follows: justification, explanation, systematization, discovery, and communication. Hanna & Jahnke (1996) discussed the relationships between mathematics and empirical sciences, and identified other functions. Recently, de Villiers (1999) added another function "proof as a means of intellectual challenge" (p.8).

The "discovery" function means that we can make new results with mathematical proof. The results include propositions, proofs, assumptions, concepts, counterexamples, definitions and so on. There are crucial interactions among these results. For example, making a new proof with the previous proof often reveals tacit assumptions, and overcoming a counterexample may lead to generalize naive concepts adopted in the previous proposition and/or proof.

In the history of mathematics, the "discovery" function has taken a crucial role to establish the realm of mathematics. For example, the concept of uniform convergence was generated as a hidden lemma that Seidel found in analyzing Cauchy's primitive conjecture "the limit of any convergent series of continuous functions is itself continuous" and proof. This generation process went through Abel's restriction to power series, the official recognition of Fourier's trigonometrical series as a counterexample, and the decline of infallibilism (Lakatos, 1976, pp.127-141).

3.2 Why the "discovery" function should be focused on?

3.2.1 Teaching and learning a proof can reflect a principle of intellectual developments in mathematics

We can find a lot of warnings in company with teaching and learning a body of knowledge systematized in science and mathematics as it is. Especially, Freudenthal (1971) distinguished mathematics as a prefabricated system with mathematics as an activity, and pointed out that the latter was more essential for a person who would like to apply mathematics, and that in order to create one's own mathematics by oneself we had to improve our teaching aims, contents, methods, and so on.

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Mathematics as an activity does not always content with the "realm" of completeness, but falls into the "mud" of fallibility frequently. However, Even if a proof confronts counterexamples, it does not mean that the proof is of no value. Rather, the proof can propose an opportunity of making new elaborated theorems. Furthermore, it is possible that assumptions hidden in the proof are revealed, the logical relationships among assumptions are adjusted, and a lot of propositions deduced from them are systematized. Thus, the "discovery" function of proof works a principle of intellectual development in mathematics.

In lower secondary school of Japan, it seems hard to find classroom activities where students take a step forward to applying the "discovery" function. If it would be applied, students could have a time to look over their proofs, and could have a chance to make new results with their proofs. Such activities may lead to their selection of assumptions and their local organization of propositions that we can see in "mathematics as an activity".

3.2.2 Applying the "discovery" function can improve students' undesirable ideas of school mathematics

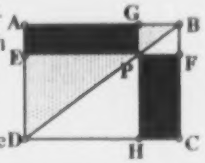
Most teaching of mathematical proof have been inclined to emphasize a feature of deduction that can establish a universality of proposition, in contrasting with the shortage of induction. However, lower secondary school students easily recognize the universality by means of induction. Thus, for students, deductive proof is nothing but a mean of justification as the same as inductive activities, although deduction only can establish a universality of proposition logically. Furthermore, it turns out to be more difficult to emphasize the superiority of deduction over induction concerning a universality of proposition, since the applications of computer technologies strengthen the accuracy and easiness of more complicated experiments. The extreme emphases of deduction will lead students to learn a mathematical proof as a "legal ritual" only in school mathematics.

We can focus on alternative features of mathematical proof that are effective for school mathematics. One of the features is the "discovery" function of proof, which has a possibility to improve students' ideas of school mathematics as follows. Concerning their ideas of mathematics, the "discovery" function of proof encourages students to recognize that mathematics never have nothing to do with me, rather it can be fundamentally constructed and established thorough their active workings. For, students will have special experiences to make their own propositions and/or proofs with their current proof, to reveal implicit assumptions, to elaborate definitions and/or concepts, and so on. Concerning their ideas of learning a proof, the "discovery" function of proof encourages students to recognize that the construction of mathematical proofs must not be reduced to the goal of learning a proof, rather it should be appreciated on as starting point. For, students will have the previous special experiences only through looking back their proving activities. Concerning their ideas about the effectiveness of school mathematics, the "discovery" function of proof will be useful for students to behave productively with intellectual honesty even in the non-mathematical situations. For, the results from deductive proof can be more certain than the inductive results and can contribute to deal with counterexamples and refutations appropriately.

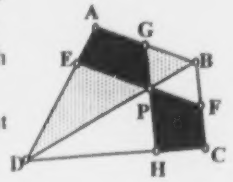
4. Methods of experiments

The experiments were carried out in a public lower secondary school located in an urban area of Nagano-city on March 31, 1998. The cooperators are eight high-achieving 8th graders. They were divided into two groups on the basis of their mathematics teachers' idea. One group consisted of four boys, and the other consisted of four girls.

The experiments included two parts of activities. In the former part, each student challenged PROBLEM 1 on a worksheet individually for about fifteen minutes with a geometrical computer tool "Sketchpad". All of them finished to make their own proofs within fifteen minutes. In the latter part, each group challenged **PROBLEM 2** on a worksheet with a geometrical computer tool in cooperation. There was no limit to the time for the latter part.



PROBELM 1 Place an arbitrary point P on the diagonal BD of rectangle ABCD. Draw a parallel line to segment AB through P, and let the intersection point with segment DA be E, and another intersection point with segment BC be F. Draw another parallel line to segment BC through P. Let the intersection point with segment AB be G, and the other intersection point with segment CD be H. Prove the area of □AEPG is equal to the area of □CFPH.



PROBLEM 2 Draw a segment BD in □ABCD, and place an arbitrary point P on the segment. Draw a parallel line to segment AB through P, and let the intersection point with segment DA be E, and another intersection point with segment BC be F. Draw another parallel line to segment BC through P. Let the intersection point with segment AB be G, and the other intersection point with segment CD be H. What conditions of □ABCD do make the area of □AEPG equal to the area of □CFPH? Prove it.

5. What are essential to apply the "discovery" function of proof?

5.1 The quality of an initial problem and its proofs

An initial problem and its proofs need to satisfy the following conditions.

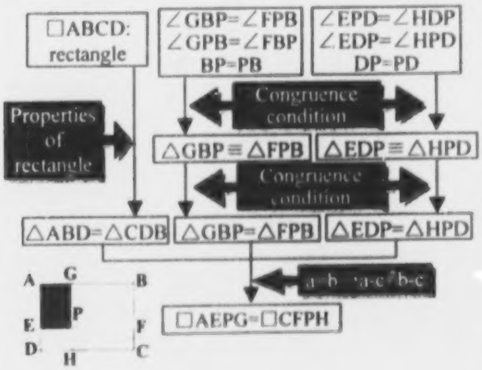
(1) An initial problem requires students to make a proof.

(2) The proof can enlighten students on the deductive configuration from assumptions to conclusions.

Concerning (1), in order to apply the "discovery" function of proof, students do not only follow the thread of a given proof, but also need to know why the thread was required. It is for that purpose that each student makes a proof on him/her own.

Concerning (2), the proof of initial problem not only verifies the proposition to be concerned, but also needs to enlighten students on the reason why the proposition is true (de Villiers, 1998, p.379). For, knowing the reason requires the deductive configuration from assumptions to conclusions, and the configuration make it possible to make new propositions and their proofs on the basis of the current proof of initial problem.

For example, a girl KYOKO made a proof of PROBELM 1 within fifteen minutes. The following was the deductive configuration of her proof. She had already learned the theorems (e.g. congruence condition) used in her proof, and it was easy to recognize the deductive configuration of her proof visually. Therefore, she could know why "□AEPG=□CFPH" was true. In challenging PROBLEM 2, she checked with measurement functions of "Sketchpad" that "△ABD=△CDB" as the condition of □ABCD led to "□AEPG=□CFPH". However, "△ABD=△CDB" could not always hold "△EDP≡△HPD" true. Then, despite her experimental justification, she changed the condition of □ABCD from "△ABD=△CDB" into "△ABD≡△CDB" in order to hold "△EDP≡△HPD" true in the same way as her proof of PROBLEM 1, and proved that "If △ABD≡△CDB in □ABCD, then □AEPG=□CFPH".



5.2 The quality of an additional problem proposed after making proofs of initial problem

It is not easy for lower secondary school students to make new results with mathematical proofs. For, it requires different ways of thinking from the way to make mathematical proofs. Additionally, in the usual teaching and learning, students have less experience to make new results with proofs. Therefore, it is necessary for students to propose an additional problem after making proofs of initial problem. The additional problem needs to satisfy the following conditions.

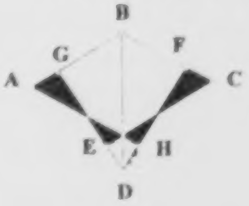
(a) The additional problem encourages students to reflect on the deductive configuration of initial proof.

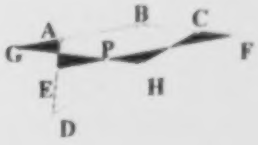
(b) The additional problem leads students to reveal tacit assumptions, to make new mathematical concepts, to prove necessary propositions as new theorems, and so on.

Concerning (a), making new results with mathematical proofs requires the deductive configuration of initial proof. Accordingly, the additional problem requires encouraging students to reflect on the deductive configuration. If students can find invariable features between the initial proof and the additional problem, they seem make great efforts to reflect on the deductive configuration. In general, we can make an additional problem as follows: the additional problem keeps conditions of the initial problem except the supposition "P" of initial proposition "P→Q", and it requires students to find alternative suppositions "R" from which they seem deduce the same conclusion "Q", and to prove the proposition "R→Q". For example, PROBLEM 2 keeps the conclusion "□AEPG=□CFPH" and construction procedures except rectangle □ABCD, and requires students to find conditions of □ABCD from which they seem deduce the same conclusion "□AEPG=□CFPH", and to prove that if □ABCD satisfies the conditions, then □AEPG=□CFPH.

Concerning (b), the "discovery" function of proof in mathematics assists to reveal tacit assumptions, to construct new mathematical concepts, and so on. Furthermore, it often contributes to prove the unproved propositions. If school mathematics should reflect these mathematical activities accompanying with the "discovery" function, the additional problem can lead students to reveal tacit assumptions, to construct new mathematical concepts, to prove necessary propositions as assumptions, and so on.

For example, in PROBLEM2, students will find "△ABD≡△CDB" or "△ABD=△CDB" as a condition of □ABCD. A kite satisfies the former condition, but □AEPG and □CFPH will disappear depending on the position of point P on BD. In order to keep these figures at any case, it is sufficient to let the intersection points E,F,G,H not with each side of □ABCD, but with each line including each side. In addition, if students find the area relation "□AEPG=△ABD-△GBP-△EDP", they have a chance to construct more advanced concepts of area as follows: "The shape has the area of 0 units despite of seeing it in the figure, "The shape has the area of negative units". (These concepts are corresponding to "proof-generated concept"(Lakatos, 1979).) On the other hand, in the latter condition "△ABD=△CDB (area equivalence)", the relation "△EDP=△HPD" can be proved by means of the height ratio of △ADB to △EDP and the height ratio of △CDB to △HDP on the two basis DB and DP, although it cannot be proved by means of the congruence of triangles which was useful in PROBLEM 1. Thus, students may prove the following proposition that is unknown for them: "If two pairs of triangle on the same base has the same ratio of similarity, and the areas of one pair are equal, then the areas of another pair are equal".





5.3 The quality of individual activities to solve an additional problem

Students' activities have the individual aspect and the social aspect, both of which are supplementary each other. This study focuses on the individual aspect. For, this aspect has more specific characteristics to the discovery function of proof than the social aspect.

The individual activities to solve an additional problem need to satisfy the following conditions.

(A) A student makes new propositions with experiments and/or initial proofs, and improves or changes them during the proving process.

(B) A student acquires the temporary configuration of new proof by means of keeping invariable parts and eliminating variable parts in the deductive configuration of initial proof.

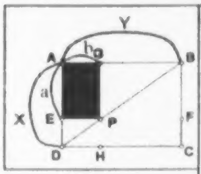
(C) A student decides whether to re-apply the already applied theorems within the initial proof in order to supplement the incomplete parts in the temporary configuration of new proof.

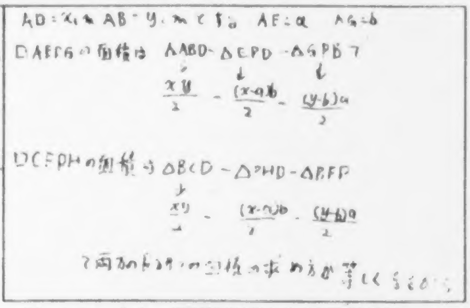
(D) In case of not re-applying the already applied theorems, a student reconstructs alternative chains of deductive reasoning from/to invariable parts.

(E) In case of re-applying the already applied theorems, a student supplements the temporary deductive configuration with alternative appropriate relations and/or theorems.

The reason of (A) is as follows. With experiments and/or initial proofs, students make new propositions that help students to decide what to assume and to conclude. Naturally enough, students will improve and/or change propositions during their solving activities of an additional problem as necessary.

For example, in PROBLEM 1, a boy TETSUYA wrote four letters (X, Y, a, b) into the given figure, and made his proof as follows. (The last line of his proof means, "Both ways of area calculation are equal".) His proof has the deductive configuration as follows.




AD=x、AB=yとする。 AE=a AG=b

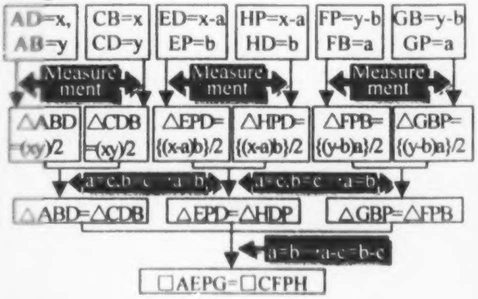
□AEPGの面積は △ABD−△EPD−△GPBで

$\frac{xy}{2}-\frac{(x-a)b}{2}-\frac{(y-b)a}{2}$

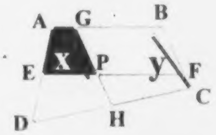
□CFPHの面積は △BCD−△PHD−△BFP

$\frac{xy}{2}-\frac{(x-a)b}{2}-\frac{(y-b)a}{2}$

2通りのやり方の面積の求め方が等しくなるので



Next, in solving PROBLEM 2, TETSUYA induced the following proposition with other three boys by means of a geometrical computer tool: "if △ABD=△CDB, then □AEPG=□CFPH". He could not represent the areas of triangle (e.g. △ABD) with some letters in the same way of PROBLEM 1. Then, with focusing on that both △ABD and △CDB had the same base BD, he represented the height of △ABD with a letter "x" and the height of △CDB with "y", and improved the proposition as follows: "If x=y, then □AEPG=□CFPH".



The reason of (B) is as follows. The deductive configuration of initial proof involves variable parts and invariable parts between an initial problem and an additional problem. It also shows why and how to apply theorems. Therefore, students need to acquire the temporary configuration of new proof by means of keeping invariable parts and eliminating variable parts in the deductive configuration of initial proof

For example, in solving PROBLEM 2, TETSUYA seemed eliminate the relations related to four letters (X, Y, a, b) in the deductive configuration of initial proof. Conversely, he kept the invariable relations of area equivalence between triangles. On the other hand, in his initial proof, he applied three theorems in the

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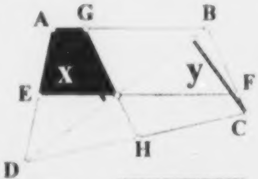
following order: "area formula" → "a=c, b=c→a=b" → "a=b→a-c= b-c". He kept this application order, since the order seemed so effective to solve PROBLEM 2. Thus, he acquired the temporary deductive configuration of new proof.

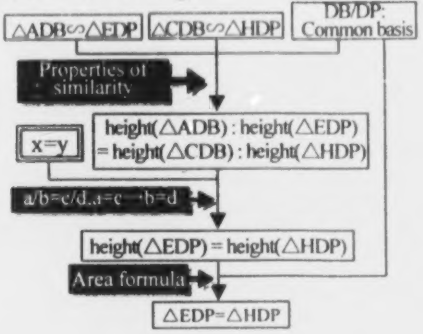
The reason of (C) is as follows. There are two ways to supplement the incomplete parts in the temporary deductive configuration. One is the re-application of the already applied theorem within the initial proof. Another is the application of other theorems. Then, students need to decide whether to re-apply the already applied theorems or not. For example, the boy TETSUYA seemed decide to re-apply the already applied two theorems "area formula" and " a=c, b=c→a=b" in order to deduce "△ABD=△CDB " and "△EPD=△HDP ". On the other hand, he seemed decide not to re-apply them in order to deduce "△GBP=△FPB ".

The reason of (D) is as follows. In case of not re-applying the already applied theorems, in order to deduce the invariable parts, and/or to deduce something from these parts, students need to choose alternative appropriate theorems. And, in order to apply the chose theorems students need to find alternative appropriate relations. Thus, students need to reconstruct alternative chains of deductive reasoning from/to invariable parts. For example, in order to deduce the invariable relation "△GBP=△FPB", TETSUYA decided to apply alternative theorems "Condition of triangle congruence", and "Properties of congruence", since □GPFB was no longer rectangle in PROBLEM 2. And, he reconstructed a chain of deductive reasoning to the invariable relation "△GBP=△FPB" from the alternative relations (e.g. ∠GPB=∠FBP) that were not used in his initial proof.

The reason of (E) is as follows. In case of re-applying the already applied theorems, students need to find alternative appropriate relations to re-apply them. The application of the already applied theorems into the alternative relations often lead to produce additional relations not be used in the initial proof. Furthermore, the application of the additional relations often leads to encourage applying unapplied theorems. Therefore, with these relations and theorems, students need to supplement the temporary deductive configuration.

For example, in order to deduce the relation "△ABD=△CDB " TETSUYA let the height of △ABD be "x", and the height of △CDB be "y" in order to apply area formula of triangle as described above. And, he introduced the relation "x=y" analytically to deduce the relation "△ABD=△CDB ". On the other hand, Concerning the deduction of the relation "△EDP=△HDP ", the relation "the height of △EDP is equal to the height of △HDP " makes it possible to apply the already applied theorem "area formula of triangle", since both triangles has the same base DP. But, the relation is not self-evident. Actually, Two other boys asked TETSUYA, "Why these height of triangles are equal?", then he explained as follows. Because of the relations "△ADB∽△EDP ", "△CDB∽△HDP ", and "DB/DP: common basis of triangle", the similarity ratios of two pairs of triangles are equal. Then, the ratios of height of two pairs are equal because of the properties of similarity. (height(△ADB) : height(△EDP)= height(△CDB) : height(△HDP)) Then, because of the relation "x=y", the height of △EDP is equal to the height of △HDP. Thus, he applied the unapplied theorems "Properties of similarity" and " a/b=c/d,a=c→b=d", and deduced the relation "△EDP=△HDP ".





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6. Conclusions

This study proposes the following conclusions.

The followings are essential to apply the "discovery" function of proof in lower secondary school mathematics. (See 5.1, 5.2, and 5.3 in detail.)

- Two conditions of an initial problem and its proofs (1), (2)
- Two conditions of an additional problem proposed after making proofs of initial problem (a), (b)
- Five conditions of individual activities to solve an additional problem (A), (B), (C), (D), (E)

These essentials will be useful to make guidance plans and teaching materials for the lessons applying the "discovery" function of proof. In order to realize the class these essentials suggest the following prerequisites.

- Teachers recognize a proof as the starting point of teaching and learning.
- Students investigate the relative relations between assumptions and conclusions through looking back his/her proof and proving process (e.g. "Which relations are indispensable to deduce conclusions?", "Without applying the relations, why the conclusions cannot be deduced?").

There are two kinds of questions worthy of future research.

- Are there other essential conditions to apply the "discovery" function of proof in lower secondary school mathematics?
- What are essential to reveal tacit assumptions and/or to construct new concepts?
- What are essential to advance toward the local organization or systematization of propositions through applying the "discovery" function of proof?

Acknowledgements

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[*] E-mail: mmiyaza@edu.shinshu-u.ac.jp, Telephone: +81-26-238-4105 (Office), Fax: +81-26-238-5540 (The general affairs section) Homepage: http://math-edu01.shinshu-u.ac.jp/General/Staff/Miyazaki/miyazaki-e.html

The Anatomy of an 'Open' Mathematics Lesson

Ida Ah Chee MOK, The University of Hong Kong

Abstract

A grade-4 mathematics lesson working on an open-ended problem in Shanghai was analysed from a perspective of learning which learning is seen as a change in a person through experience. Whether one can discern the critical aspects in the object of learning or not depends on variation. Thus, the analysis aimed to describe how the space of variations was created when the teacher adopted an 'open' pedagogical approach in teaching.

Introduction

Problem solving has a long established status in mathematics education. More recently, educators in China have developed a growing interest on the use of open-ended problems in the teaching of mathematics. Besides looking closely at the nature of a wide range of problems, the adaptation of an open pedagogical approach is also fostered inside the classroom. The lesson in the current paper was a demonstration lesson in the National Conference for "Open-ended Questions" which took place in Shanghai in November 1998. The lesson was 'open' in a dual sense that it focused on an open-ended problem and that the teacher had adopted an open pedagogical approach. Despite the overwhelming compliments on the general aspects of the lesson, the question, "What makes the lesson a good lesson?" has not been properly addressed. In brief, the lesson had left a good impression but it was difficult to tell what the students had learned. It is not difficult to anticipate such a conflicting sentiment as learning problem solving is unlike the learning of isolated skills. In this paper, I choose another point of departure, proposed by Marton and Booth (1997) that learning is a way of experiencing. The lesson in Shanghai could be seen as a mediated experience of problem solving. The next section will summarize this perspective of learning as a backdrop. It is followed by an analysis of the lesson by applying the key concepts in the framework, namely, variation and simultaneity. Finally, the discussion will look into how the study contributes to an understanding of an open pedagogical approach in an open mathematics lesson.

Theoretical framework

Attempting to describe the world as it is seen from the point of view of the learner traces its origin in Phenomenography (Marton, 1981). In phenomenographic research the object of analysis is the variation in which people conceive of, understand or experience something. During last twenty-five years, there has been significant development in this specialization of research. Studies have revealed a variation in the way which various phenomena such as number, Newtonian motion, or learning are experienced by the learners.

How a learner experiences something depends on the structure and organisation of the learner's awareness. The nature of human awareness is complex and dynamic. In slightly different words, experience is what is discerned in action and learning is a change in a person through experience. For example, in order to learn what a triangle is, the learner must experience different triangular and non-triangular shapes. Variations with a focus on the important aspects such as the number of sides and angles must be experienced either independently or simultaneously in different examples. It is in the experiencing of the variations between different triangles and those between triangles and non-triangles, the learner develops the capability of discerning between triangles and non-triangles. Thus, variation is pertinent to learning and we would argue also to teaching.

As variation, simultaneity and discernment are critical aspects of learning, it will be reasonable to apply these concepts to study teaching. More recently, there have been attempts which investigated teaching from this point of departure. Runesson (1997,1999) used these concepts for analyzing mathematics lessons on the topic of rational numbers in Sweden. She has shown that teachers directed their students' awareness to more than one aspects simultaneously by opening dimensions of variation. For example, a teacher in Runesson's study used a rubberband divided into four parts. By stretching the rubberband alongside an object, a quarter of the length of the object could easily be found. By "measuring" different objects of different length with the rubberband, the teacher varied the absolute sizes of the whole. Consequently, the absolute size of a quarter varied, whereas the relative size (1/4) was kept constant. So in this case, the absolute sizes of the whole as well as a quarter (i.e., which both vary) constituted dimensions of variation in the teaching process. Moreover, the teacher represented ¼ by manipulative aids, she talked about "a quarter" and she wrote it on the blackboard with symbolic representations. That is, the teacher also opened a dimension of variation in the representation of ¼. Runesson explains that in this particular situation, there are simultaneously (at least) three dimensions of variations present in the teaching process. There is a variation in the representational mode, in symbolic form and a variation in sizes of the whole that is partitioned. Such dimensions of variation that are opened up, are dynamic and sometimes overlap each other. When different dimensions of variation are opened up in the teaching situation, a space of learning is constituted.

In the next section, the concepts of variations and simultaneity will be used to analyze a mathematics lesson in Shanghai. Besides looking for the dimensions of variation, the pattern of variations and how these variations were created are important in the analysis.

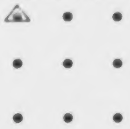
Starting from a real life context

The class was primary-4 (grade 4) working on a problem about a postman's route. It was a 45-minute demonstration lesson. There were 28 students and they sat in groups of 4.

To begin, Miss Zhu, the teacher, explained the problem by holding up a sample worksheet (figure 1).

> There are nine dots on the paper. The dot at the left upper corner, surrounded by a triangle represents the post office. The postman needs to start from the post office, send a letter to each of the eight places and return to the post office. What could be the postman's route?

Figure 1 The worksheet

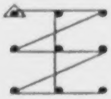


The teacher asked the students to work in groups and try to *design as many routes as possible.* Each student then designed his/her own routes on the pieces of paper and put all the designs in the group in a pile. They were very efficient and only looked at their neighbour and talked occasionally. When they finished all the paper on their table, they raised their hands for more. After the teacher had resumed the attention of the whole class, she posted the results on the blackboard and asked the class to judge whether or not the designs were correct. This group produced 18 designs. The teacher said that the students were allowed to discuss with group members. Some rustling was heard and the whole class agreed that all the designs were correct.

In this part of the lesson, the students created a lot of possible paths (on pieces of paper) some of which were posted on the board and formed the bases for later development of the lesson. The open nature of the problem thus created a dimension of variations between many possible drawings.

While the teacher invited the class to inspect the accuracy of the paths posted on the board, she deliberately focused on a wrong design, which was from another group, and asked for justification (see figure 2). In order to discern between correct and incorrect paths, a student pointed out that the arrows were missing and another student pointed out that the path did not return to the post office.

Figure 2 The wrong route



Thus, this search for justification constituted a dimension of variation. Then, the teacher raised the question which type of design would be the best. This question again made up a dimension of variation as there could be different criteria for the best routes. The students' suggestions were "the nearest path", "non-repeating". In addition, a student expressed her concern for the case of an urgent letter. After a few exchanges of ideas between the class and the teacher, they agreed that the nearest route was the best.

T: How is the best design?

S3: Use the nearest path?

T: 'Use the nearest path', good. Any others?

S4: Don't repeat the path.

T: Good, 'don't repeat the path'. Any others?

S5: If there is an urgent letter which need to be delivered first, what should we do?

T: Good, 'if there is an urgent letter'. Very good. Any others?

(No more suggestions.)

T: Let's first put this problem aside and assume that there are no urgent letters. All letters are the same (have the same priority). Then, how is the best design?

S6: The nearest route.

Although the design of possible paths and the discussion were both embedded in the postman's problem, the teacher deliberately directed the students to reflect upon either the postman in the real life context or their own drawings/patterns. The former referred to designing as many possible routes as possible and finding the criteria of the best routes, whereas the latter referred to distinguishing between the correct and wrong routes and picking up the best routes. This variation (between focus on the postman and the patterns) was of a higher order. Furthermore, the students' acts demonstrated another type of variation between creating (drawing many possible paths), comparing patterns (distinguishing the correct path from the incorrect), reasoning (giving criteria for the best routes), and back to analysis of patterns (searching for the shortest routes). For the first three, the students' answers were in fact situated in the real life context. It was only in the last part (i.e., picking the nearest routes), that the students began to engage themselves in a mathematical context of studying patterns. The context of the problems/activities thus constituted a dimension of variation between real life and mathematical contexts.

Working in a mathematical context

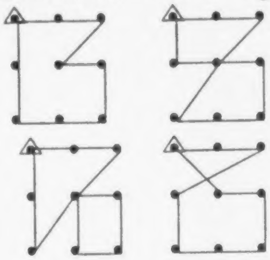
The activities following this part were then entirely embedded in a mathematical context with a focus on patterns. First, the teacher intervened by guiding directly the class to compare the numbers of straight (horizontal and vertical) and diagonal segments in the patterns. They worked out the numbers for four patterns (see figure 3). Then, the teacher asked why one of the routes were the shortest. The four patterns and the additional strategies created a new open mathematical problem in which the students needed to argue for the relationship between the patterns and the number of line segments. As different students gave different answers, we can see that this open problem, which focused on the relationship between the patterns, constituted a dimension of variation. These students' reasons were used in the later part of the lesson to sort out the shortest routes from the massive number of patterns.

S8: The first has one diagonal line segment less than the others.

S9: The first has only one diagonal line segment and the sum of the two is nine. The number here is the least.

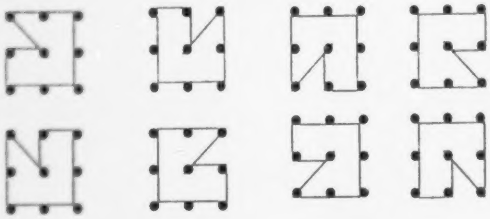
Figure 3 The four patterns and the number of line segments

Straight segments	Diagonal segments
8	1
8	2
8	2
7	2



The teacher removed the designs of the longer routes from the blackboard, leaving seven on the board. Meanwhile, the students did the same with their own drawings in their groups. Then, the teacher asked students to supply more designs of the shortest route from their own piles. The students handed in five more to make a total of twelve on the blackboard. Then the teacher suggested the class to neglect the arrows. As a result, there were only eight patterns left on the blackboard and they set the scenario for the next mathematical problem of categorizing the patterns (see figure 4).

Figure 4: The 8 patterns after neglecting the arrows



This enunciation of the categorization problem was followed by a very active whole class discussion. A student said that there were 8 types according to the openings of the patterns. When the teacher asked for more suggestions, another student suggested to move the second one on the second row to become the third one. At this point, the students' approaches help us see how the catergorization problem could make up a dimension of variation. The former approach was a categorization of static patterns and the latter was based on transforming a pattern in order to look for a matching image. The latter idea started off a lot of suggestions how to transform (rotate or turn (reflect)) the patterns in order to put transformational images into the same category. Each single transformation then made up a dimension of variation between spoken, enactive and symbolic forms as it was first described by the students orally, the drawing was moved by the teacher, finally represented by written symbols on the board produced by the teacher. The beginning of the discussion was as below.

> T: Now, we do not consider directions. Look at these shapes. I would like you to use different methods to categorize.
>
> S11: There is eight types. The direction of the opening gives four categories. There are two types for each category. Eight altogether. (See figure 4)
>
> S12: Move the second on the second row around, then it becomes the third on the second row. (The teacher then moved the pattern according to the student's instruction.)
>
> T: Very good. She found that this pattern, after rotating, will then become the same. Any more?
>
> S12: Move the third on the second row then become the third on the first row.
>
> T: Teacher first labels these patterns. One, two, ..., eight (students echoed the numbers and the teacher wrote the numbers under the patterns). Lee said that the six rotated and became the eight. What next?
>
> T: (A student said something softly.) Fung said that rotated again to become seven. (The teacher wrote "6-8-7" on the board.)

The students suggested altogether 18 rotations and 7 reflections. In the later part, some students withdrew their suggestions immediately after they had noticed the repetition. Thus, when they were seeking new transformations of patterns, they also checked their own suggestions against the collective findings by their peers. For example,

> S25: Turn the eighth upside down, then it becomes the second. Turn the fourth upside down. Oh, it was already there.

Here, we observed that embedding in the categorization of patterns, there was a variation between the perspectives of static patterns and transformations. Within the transformation of patterns, there were the variations between rotation and reflection. Moreover, the representation of each rotation / reflection was varied between the spoken form by the student, the enactive form by the teacher / student (as they moved the patterns on the board according to the students' suggestions), and the symbolic form by the teacher.

When the students' suggestions were gradually decreased and it approached the end of the lesson, the teacher asked a new question which embodied all these observations. She requested students to reflect upon the relationship between patterns, asked for the number of categories. In the last part of the lesson, different students had different ideas about the number of categories such as, infinity, two, eight and sixteen and they supported their answers with their own mathematical reasons.

The analysis of the lesson have come to an end. To conclude, we find seven types of variations:

1. context (real life / mathematical),
2. different open questions or tasks (each elicited a range of different answers from students),
3. focus of questions or tasks (postman / patterns / relationship),
4. students' engagement in different problem solving processes,
5. different methods of categorization (static shapes / transformation),
6. different transformations (rotation / reflection),
7. different representations of a specific transformation (spoken / enactive / symbolic).

The first four levels happen to be simultaneous and link all the variations and events in the lesson. The last three are hierarchically embedded in a mathematical context and again simultaneous. It is this simultaneity which demonstrate how the space of learning is constituted as a coherent whole.

What had the students experienced?

Via the analysis of the lesson, we see that the class was engaged in a real life context about the postman's problem, shifting into a mathematical context of categorizing patterns, then back in the real life context again. In the lesson, we see examples from more than one mathematical topic: (1) the solving of the postman's problem was an example of mathematical modeling which introduced the shift from the real life context to the mathematical context; (2) the search for the best routes was an optimization problem; and (3) the categorization of patterns was in fact geometry. It is tempting to say that an objective was to teach all these topics. If so, we can easily criticize that none of these topics had been dealt with properly in terms of depth and clarity. However, if we refer to the teacher's original plan, only problem solving was seen as the objective, all these topics (modeling, optimization, and geometry) were not mentioned. How could this be the case? A simplified answer is that a problem serving good pedagogical values need to provide opportunities for students to experience mathematical concepts and power (modeling, optimization and geometry in this case) in a skilful way and room for exploratory work.

In the lesson, the teacher played an important guiding role by posing a series of sub-problems, which appeared in the form of open-ended questions or tasks. Such close monitoring had prevented the students falling into a wild-goose chase which was very common for novice (Schoenfeld, 1992). When the teacher changed the focus from "as many routes as possible" to "the shortest routes", she let students experience the *different ways of approaching a problem*. As each open-ended question made up a dimension of variation, it also provided an opportunity of *exploring the mathematical objects in focus*. The questions built upon the earlier collective outputs by the class, thus appeared naturally in a coherent sequence. Simultaneously, the foci of sub-problems varied between the postman, patterns and relationship. Consequently, the students, guided by the teacher, had visited the idea of modeling, followed by optimization and geometry. As planned by the teacher, the activity of categorizing patterns required the students to raise their inquiry to a further level. Putting the postman's problem aside temporarily, transformation was implicitly brought into focus in the class discussion. It was implicit because the term "transformation" was never mentioned in the lesson. Nevertheless, the representations of the concept varied between the symbolic, enactive and symbolic forms. These mathematical objects (transformations) thus became concrete and were played by the students explicitly and collectively in the class discourse.

All these mathematical explorations were not simply a mathematician's game. The lesson was carefully planned and carried out by the teacher. Each open-ended question created a dimension in which the teacher and the students jointly created the variations. Comparing the students' suggestions elicited by teacher at the beginning of the lesson with those found near the end of the lesson, the latter had go beyond naïve intuition and towards a critical mathematical analysis. This transcendence was accomplished in a mediated experience.

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INTERVIEW-BASED ASSESSMENT OF EARLY MULTIPLICATION AND DIVISION

Joanne Mulligan, Macquarie University, Sydney, Australia
Robert (Bob) Wright, Southern Cross University, Lismore, Australia

This paper describes the integration of multiplication and division strategies within a research-based learning framework in number. The framework, consisting of five levels of multiplication and division knowledge, is described in order of increasing sophistication from initial grouping and perceptual counting to abstract composite units and repeated addition and subtraction, and to multiplication and division as operations. An interview-based videotaped assessment of 5 to 7 year old children is described as part of an extensive professional development numeracy project in Australian schools.

In Australia, a research-based, system–wide numeracy project 'Count Me In Too', has extended a learning framework in number in order to focus on 5-7 year old students' early muliplication and division knowledge. This work draws extensively on research in this area (Greer, 1994; Mulligan & Mitchelmore, 1997; Steffe, 1994; Vergnaud, 1992; Wright,1998). The project adopts a school-based professional development model where teachers and mathematics consultants collaborate in assessing and analysing students' videotaped interviews (NSW Department of Education and Training, 1998; Wright, Martland & Stafford, 2000). This paper describes the links between levels of development in multiplication and division and key assessment tasks. These levels and assessment tasks are central to cits focus on promoting increasingly sophisticated multiplicative strategies in young children.

Background

Studies investigating multiplication and division processes with younger children have identified the development of sound problem-solving strategies from an early age and the importance of modeling and representation in this development (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997; Steffe, 1994). In longitudinal analyses of young children's intuitive models for multiplication and division, increasingly sophisticated strategies based on an equal-groups structure and calculation strategies were described. In these analyses counting strategies were integrated into repeated addition and subtraction processes and then generalised as the binary operations of multiplication and division. Strategies used with concrete and sensory models were internalised and replicated at an abstract level with increasing sophistication.

The development of composite structure

Children's early multiplication and division knowledge results from cognitive reorganisations of their counting, addition and subtraction strategies, and builds on number word sequences, combining and partitioning. However it differs from addition and subtraction mainly because the former incorporates the ability to use equal groups as 'abstract composite units' (Steffe, 1992):

> An abstract composite unit [is] the result of applying the integration operation to a numerical composite or to a symbolized numerical composite. The child focuses on the unit structure of a numerical composite e.g. one ten, rather than on the unit items e.g. ten ones (Steffe & Cobb, 1988, p. 334).

A developmental framework describing the growth of multiplication and division processes must be based on the acquisition of an equal-grouping (composite) structure (Mulligan & Watson, 1998). A composite whole is a collection or group of individual items that must be viewed as one thing. For example, a child must view three items as "one three" in order for the unit "three" to be a countable unit. For an advanced understanding of multiplication and division the child needs eventually to co-ordinate groups of equal-sized groups and to recognise the overall pattern i.e. composites of composites, e.g."three sixes". Steffe (1994) describes this as a premultiplying scheme:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit (p.19).

Once the initial elements are developed and consolidated with repeated addition or repeated subtraction and sharing models, understanding must extend beyond these to a point where the commutativity of multiplication is recognised and the inverse relationship between multiplication and division is applied. The development of multiplication and division as inverse processes forms the basis of a developmental model of composite structure. The acquisition of multiplication and division as binary operations relies on the child's ability not only to develop composite structure and commutativity but also to recognise the relationship m x n where m is the composite unit 'operated upon' n times. This is quite different to a repeated addition notion of multiplication.

Children may use identical or similar strategies for solving both multiplication and division tasks except that in division, the child will form and count composite units from a known quantity. Interestingly, it has been found that division is not necessarily more difficult than multiplication, and in many situations, division situations may be easier than multiplication. For example, it may be easier for a child to share counters into equal groups and count the number of groups rather than keep track of a larger number of composite groups for multiplication. Teaching children to share and group small numbers into equal parts can facilitate the development of multiplication and division strategies i.e. non-count-by-ones strategies (Mulligan & Mitchelmore, 1997).

Table 1 outlines a progression of five levels in children's development of early multiplication and division knowledge. A key distinction is made between levels 2 and 3 where children progress from using visible or sensory items to where items are partially or fully screened. (Examples of videotaped excerpts support the classification of strategies into levels).

Table 1 Development of multiplication and division strategies

Level 1 Initial grouping and perceptual unitary counting: models or shares by dealing in equal groups but they do not see the groups as composite units; counts each item by ones (perceptual).

Level 2 Perceptual counting in multiples: models equal groups and counts using rhythmic, skip or double counting; counts the number of equal groups and the number of items in each group at the same time only if the items are visible.

Level 3 Figurative composite units: models and counts without visible items ie the child can calculate composites when they are screened, where they no longer rely on counting by ones. The child may not see the overall pattern of composites such as "3, 4 times".

Level 4 Repeated addition and repeated subtraction: uses composite units in repeated addition and repeated subtraction. Uses a composite unit a specific number of times as a unit e.g. 3 + 3 + 3 +3; may not fully co-ordinate two composite units.

Level 5 Multiplication and division as operations: two composite units are co-ordinated; recalls or derives easily, known multiplication and division facts; uses multiplication and division as an inverse relationship.

At Level 1 the child can establish the numerosity of a collection of equal groups when the items are visible and counts by ones when doing so, that is, the child uses perceptual counting. The child can make groups of a specified size from a collection of items (quotitive division), for example given 12 counters the child can arrange the counters into groups of three thereby obtaining four groups. The child can share a collection of items into a specified number of groups, for example, given 20 counters the child can share the counters into five equal groups (partitive division). The child does not count in multiples.

At Level 2 the child has developed multiplicative counting strategies involving implicitly or explicitly counting in multiples. After sharing a collection into equal groups the child uses one of these strategies to count all the items contained in the groups which are necessarily visible. The child is not able to count the items in situations where the groups are screened. These counting strategies include rhythmic, double, and skip counting and each is given the label 'perceptual' (e.g. perceptual rhythmic counting) because of the child's reliance on visible items.

At Level 3 counting strategies do not rely on items being visible and do not involve counting by ones. For example, if the child is presented with four groups of three counters, where each group is separately screened, the child may use skip counting by threes to determine the number of counters in all, that is 'three, six, nine, twelve'. From the child's perspective each of the four screens symbolises a collection of three items but the individual items are not visible. There is a correspondence between not having to count by ones on tasks involving equal groups and counting-on in the case of an additive task, for example 6+3 presented with two screened collections. In the case of the additive task the first screen symbolises the collection of six counters and the child does not need to count from one to six.

At Level 4 the child can use repeated addition to solve multiplication tasks and repeated subtraction to solve division tasks and can do so in the absence of visible or screened items. On a multiplicative task involving six groups of three items, in which each group is separately screened, the child is aware of each group as an abstract composite unit.

The child, at Level 5, can co-ordinate two composite units in the context of multiplication or division. On a task such as six threes, or six groups of three, for example, the child is aware of both six and three as abstract composite units, whereas at Level 4, the child is aware of three as an abstract composite unit but is not aware of six as an abstract composite unit. They can immediately recall or quickly derive many of the basic facts of multiplication and division and may use multiplication facts to derive division facts. The commutative principle of multiplication (eg 5x3 = 3x5) and the inverse relationship between multiplication and division are within the child's zone of proximal development. Thus, for example, the child might be aware that six threes is the same as three sixes and might use 4x8 = 32 to work out 32÷4.

Two scenarios describing children's solutions to multiplication and divisional tasks are discussed in terms of the particular level for each scenario.

Scenario 1 – Charlotte and Anthony– Level 3

In this scenario Anthony's first two tasks are to establish the numerosities of a screened 5x3 array and a partially screened 5x4 array. Following this Anthony is presented with a divisional task involving quotitive sharing.

C: (places out a 5x3 array with one screen covering three rows and a second screen covering the other two rows. Briefly unscreens and then rescreens the three rows). Under here there are three rows of three and under here there are two rows of three. How many rows are there altogether?

A: Five.

C: How many dots in each row?

A: Three.

C: How many dots are there altogether?

A: (After six seconds) fifteen!

C: How did you work that out?

A: I said, three, six, nine, and three more makes twelve and three more makes fifteen.

C: (Places out a 5x4 array on which 12 dots in a 4x3 array are screened). I've covered part of this dot pattern. How many dots are there altogether?

A: (Looks at the array and moves his head from left to right and back five times in coordination with five subvocal counts). There is -- twenty!

C: How did you work that out?

A: I counted all the rows.

C: Tell me how you counted them.

A: (Points to each row of four in turn). I went four, eight, twelve, umm --, sixteen, twenty!

C: There are twelve biscuits and the children are given two biscuits each. How many children would there be?

A: (Places his right hand on the desk and speaks softly). Twelve biscuits --. (after 11 seconds) there is --.

C: Pardon?

A: There is 12 biscuits (pauses) and we gotta share 'em.

C: Hmm, hmm. So that they get two biscuits each. How many children would there be?

A: (Looks ahead and then quickly moves his right hand twice along the desk). One, two (subvocally. As before, quickly moves his hand twice), three, four (subvocally, and then moves his hand twice for a third time) five, six -- (subvocally. Pauses for two seconds, and then makes four pairs of two movements on the desk in coordination with counting subvocally). One, two --; three, four --; five, six; seven, eight --. (pauses for one second, and then makes three pairs of two movements on the desk in coordination with counting subvocally). One, two --; three, four --; five, six --. (pauses briefly, and then taps the desk five times in a 2-2-1 pattern in coordination with counting subvocally). One, two --; three, four --; five --. (touches the desk three times). Three children.

Anthony used skip counting and repeated addition to establish the numerosity of a screened 5x3 array and a partially screened 5x4 array. In explaining his solution of the task involving the 5x3 array he said 'three, six, nine, and three more makes twelve, and three more makes fifteen'. He similarly explained his solution of the task involving a 5x4 array. These solutions indicate that Anthony is at least at Level 3 because he solved tasks in which the items were not visible and in doing so counted equal groups by multiples. That the task involving quotitive sharing did not involve visible or screened items is significant in determining Anthony's level. Anthony did not use repeated subtraction or repeated addition when attempting to solve this task. Having done so would indicate that he could conceptualise 'two' as an abstract composite unit. That is, he could regard 'two' simultaneously as two ones and one two.

By way of contrast Anthony attempted to enact making groups of two using twelve imaginary biscuits. But he was unable to keep track of the number of groups and the number of biscuits remaining after he had enacted making three groups of two. In the absence of visible or screened items it was necessary for Anthony to attempt to enact making equal groups of two from twelve. Because 'two' was not an abstract composite unit for Anthony and because he could count in multiples to solve tasks involving equal groups he is judged to be at Level 3, figurative composite grouping.

Scenario 2 – Amanda and Joshua – Level 2

In this scenario Joshua's first task is to produce the number word sequence of multiples of three, his second task is to establish the numerosity of a 5x3 array, and his third is to establish the numerosity of a 5x4 array. On the fourth task he is asked to count the 5x4 array by counting the rows of five rather than the rows of four.

A: Count by threes.

J: Three, six, (after four seconds) nine, (after three seconds) twelve, (after two seconds) fifteen, (after four seconds) fifteen.

A: Okay stop. Thank you. (places out five rows of three counters arranged in a 5x3 array). Can you count those now?

J: (Places a finger on each counter in the first row and moves the counters) three, (similarly moves the second row) six, (moves the next three rows in coordination with the saying the number words) nine, twelve, fifteen.

A: (Places out a 5x4 array of dots). How many dots are there altogether?

J: Four, and four makes eight, nine, ten, eleven, twelve, thirteen.

A: Can you count the rows in fives the other way (indicates appropriately)?

J: Places hand on row of five. That's five, ten, fifteen, (pauses briefly) sixteen, seventeen, eighteen, nineteen, twenty.

Joshua counted a 5x3 array of counters by moving each row of three in coordination with saying the multiples of three. After unsuccessfully attempting to count a 5x4 array by fours he counted the array by fives to fifteen and continued by ones to count the fourth row of five. Because he could use multiples of three or five to count visible collections Joshua is judged to be at Level 2, that is perceptual counting in multiples. Joshua's strategies differ from Anthony's (Scenario 1) because they involved counting visible collections by multiples whereas Anthony's strategies involved counting screened collections by multiples.

The development of efficient counting (non-count-by-ones, skip and double counting) and composite units are integral to developing composite structure. Co-ordinating composite units, e.g. " three threes as a unit of 9" depends on the ability to move beyond counting based on a unitary notion and to use a pattern of multiples as a double count ("1, 2, 3 (one), 4, 5, 6 (two)" etc.) mentally. While the development of direct counting and visual modeling precedes the development of abstract composite structure there exists a complex interrelationship between counting and composite structure at the abstract level. The use of skip and double counting procedures gives rise to more efficient processes that take advantage of the equal-grouping structure where repeated addition (or subtraction) is generalised as an operation.

The development of repeated addition or repeated subtraction at Level 4 does not constitute a full conceptual understanding of multiplication or division. Level 5 distinguishes the development of multiplication and the related division process as the distribution of a composite unit across elements of another composite unit e.g. generalising the structure of composites, for example, as "six, three times as 6 x 3 = 18". Critical to developing this relational understanding of multiplication is the ability to see multiplication and division in an inverse relationship and to explain commutativity such as 6 x 9 = 9 x 6. Children who are able to simply recall multiplication and division number facts without being able to explain and represent the composite structure are not functioning at Level 5.

Implications

Although multiplication and division does not usually emerge in instructional programs until the second or third grade the 'Count Me In Too' project assesses the development of these processes in order to formulate more valid assessment techniques than traditional multiple choice tests. Teaching approaches, based on the assessment, encourage closer links between concrete and abstract thinking in order to promote increasingly sophisticated strategies.

The assessment and development of multiplication and division strategies in the early years of schooling requires professionals to integrate key aspects of developing composite structure within number learning generally. Although grouping and sharing processes may on the surface, form part of "traditional" practice, the systematic and explicit nature of the framework allows professionals to gain further insight into matching learning experiences with the child's potentialities. Assessment of multiplication and division levels has already indicated that children in years K-2 have already well developed multiplication and division strategies. The project has been implemented in over 1200 government schools in the state of New South Wales and throughout New Zealand. The extension of the framework to include fractions, measurement and geometry will reshape curriculum reform and teaching practices in the early years of schooling.

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PROCEDURES OF FINDING A SOLUTION FOR WORD PROBLEMS

A STUDY OF MOZAMBICAN SECONDARY SCHOOL STUDENTS

Adelino Evaristo Murimo, Universidade Pedagógica – Beira – Mozambique

In this paper I will discuss the results of an investigation into procedures of finding a solution for word problems among Mozambican students, and compare them to those of Australian students. MacGregor and Stacey (1996) investigated the progression of students' use of algebra in word problem solving. A sample of 90 Australian students was tested three times. In the final test the students were in Years 10 and 11 (age 15–16). In that test only 35 students (39%) attempted some algebra despite of explicit instruction to work out the answers by writing equations and solving them. I tested the same problems with 43 Mozambican students (grade 8, age 14–15) at the end of the 1999 school year. In grade 8, the first secondary school level, students have their first experience with formal algebra. In my study I found that all 43 students resorted to equations to solve the problems. In the paper I will also discuss some possible reasons why the students resorted to equations.

1. Introduction

According to Bednarz and Janvier (1996) and Rojano (1996) there are two groups of procedures for solving word problems: i) arithmetic procedures (non–algebraic procedures) and ii) algebraic procedures. In the arithmetic procedures, the student works from the known to the unknown. He creates links between known quantities and operates with them. The unknown quantity appears at the end of the process. In the algebraic procedures, the student works from the unknown to the known quantities. He starts to work with the unknown and creates links between unknown and known quantities, doing this as if the unknown were known quantities.

MacGregor and Stacey (1996) investigated the progression of the use of algebra in word problem solving. A sample of 90 students was tested three times in a period of 10–month. In the final test the students were in Years 10 and 11 (age 15–16). In that test only 35 students (39%) attempted some algebra, despite of explicit instruction to work out the answers by writing equations and solving them. For each problem, 53 students, approximately 59% of the sample, obtained the problem answers (right or wrong) by arithmetic procedures. In Mozambique, I have been working with the same grades. According to my experience, students seem to resort to equations even if arithmetic procedures are suitable for the problem. After reading the MacGregor and Stacey's paper, I felt motivated to investigate which procedures Mozambican students use to solve word problems (arithmetic or algebraic) in order to confirm my impression.

2. Methodology

2.1 The sample

The study was carried out at the end of 1999 school year, and involved 43 grade 8 students (age 14–15). The students were randomly selected in two secondary schools in Beira. In grade 8 the students have their first experience in formal school algebra. According to their textbook, the students solve a diversity of word problems.

2.2 The problems

The three problems below come from MacGregor and Stacey, and were given to Australian students. The same problems I gave individually to Mozambican students.

1) A group of scouts did a 3–day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?

2) Jorge washes 3 cars. The second car takes 7 minutes longer than the first one, and the third car takes 11 minutes longer than the first one. Jorge works for 87 minutes altogether. How many minutes does he take to wash the first car?

3) The three sides of a triangle are different lengths. The side lengths add up to 63 cm altogether. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. How long is the first side?

To characterize the problems I used the Bednarz and Janvier (1996) schema (see the diagrams in figure 1).

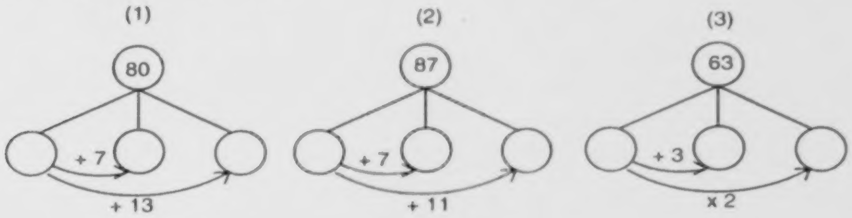


Figure 1

Structure of the problems in diagrams

The diagrams show that problems 1 and 2 involve two additive relationships and problem 3 involves an additive and a multiplicative relationship. The arrows show that the second and the third unknown of each diagram are expressed in terms of relationships and all of them are related to the first unknown. The diagrams show also that the known quantities in the problems are 80, 87 and 63.

2.3 The categories for data analysis

MacGregor and Stacey (1996) organized data into three categories: "No algebra", "Partial Use" and "Equation". By "No algebra" they mean, the student did not attempt to use any algebraic notation to solve the problems. By "Partial Use" they mean, the student attempted to use some algebraic notation, even if only to denote an unknown quantity. By "Equation" they mean, the student wrote a correct equation (equation in one variable). In my analysis I will follow these categories.

3. Results and analysis

The table in figure 2 shows the results of the problems in the categories.

n = 43	Problem 1	Problem 2	Problem 3
No algebra	0	0	0
Partial Use	27	29	23
Equation	16	14	20

Figure 2: Results of the problems

3.1 The "Equation" category

The first concern of all the students was to write the letter "x" and attempt an equation for all the problems of the test. However, only 16 students produced a correct equation for problem 1: $x+x+7+x+13=80$; 14 students produced a similar equation for problem 2: $(x+x+7+x+11=87)$. It was a little bit easier to produce the equation for problem 3: $x+x+3+2x=63$ (20 students). Although problem 3 involved the processing of two distinct operations ("+" and "x"), 4 students who did not produce a correct equation for problems 1 and 2, produced a correct equation in this problem. Maybe the geometric nature of the problem helped the students to produce the equation, or to visualize it as we will see later with the example of Julieta. Thus, there were 50 correct equations out of 129 possible in the three problems (39%).

Some students who produced correct equations did not solve them correctly. From 16 correct equations for problem 1, 11 were correctly solved. From 14 correct equations for problem 2, 9 were correctly solved and from 20 correct equations for problem 3, 11 were correctly solved.

The students' difficulties in solving the equations derived from the misinterpretation of the letter "x" and the introduction of exponential notation in the equations to reduce the number of letters (observed in only one student). This phenomenon was also observed by MacGregor and Stacey (1996) among Australian students. The major difficulty among Mozambican students was related to techniques to manipulate letters and numbers from one side to the other of the sign "=".

As examples of students' mistakes when attempting equation solving here the works of two girls:

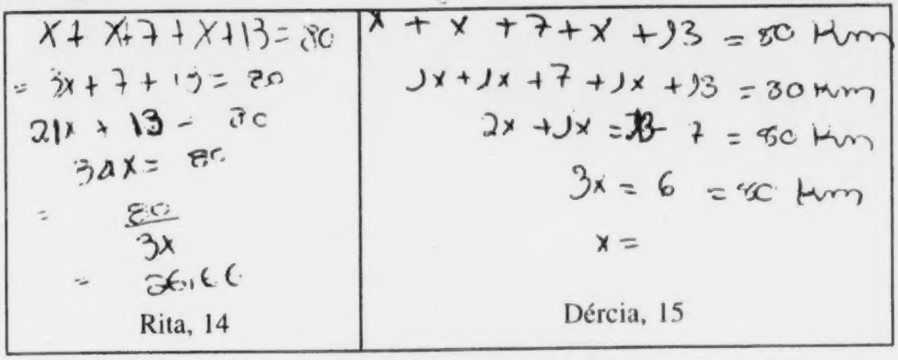

x + x + 7 + x + 13 = 80
= 3x + 7 + 13 = 80
21x + 13 = 80
34x = 80
= 80/3x
= 26,66
Rita, 14
x + x + 7 + x + 13 = 80 Km
1x + 1x + 7 + 1x + 13 = 80 Km
2x + 1x = 13 - 7 = 80 Km
3x = 6 = 80 Km
x =
Dércia, 15

Figure 3: Equation solving attempts

Rita added correctly x + x + x = 3x. Dércia transformed x into 1x. Maybe she learned it from her teacher. Myself, often heard teachers explain that the letter "x" has its number coefficient behind, and it's 1 because 1 multiplied by "x" is "x" again. The teachers use this approach to explain why x + x + x is 3x. Küchemann, (1981) observed students writing 1 behind a letter e.g. 4h + 1t instead of 4h + t. It reveals that the letter is actually used as a shorthand or as an object.

Dércia reduced 3x+7 into 21x. The number 21 comes from 3 times 7. Probably she interprets "x" as a sign of multiplication as we usual do in the classroom e.g. 3 x 7 is 3 times 7. She, however is not consistent in her reasoning because in 21x + 13 she adds 21 and 13. Küchemann (1981) also observed that students may add 21 and 13 despite of existence of x in front of 21. They add numbers that are really meaningful to them. Boulton-Lewis et al. (1996) also observed that misinterpretations of binary expressions (21x) when students start learning algebra are very common. In the equation x+x+3+2x = 63, one student left out the number coefficient (2) and continued working normally with the equation x+x+x+3+x= 63.

3.2 The "Partial Use" category

The students categorized in "Partial Use" wrote incorrect equations. There were 79 incorrect equations out of 129 equations possible in the three problems (61%). The incorrect equations written were the following: problem 1: x+7+13=80 (27 students); problem 2: x+7+11=87 (29 students); problem 3: x+3+2=63 (23 students). It is interesting to observe that all the students who wrote incorrect equations, managed to solve them; maybe because the equations contained only one "x". When the number of letters increased in an equation the students had much more difficulties solving it. It is also interesting to observe that the students who wrote incorrect equation as for example x+7+13=80 and determined x = 60 when they saw that 60 worked well in their equation by doing 60+7+13=80, were convinced that 60 was a solution of the problem, forgetting to question the validity of the equation itself.

As I said before, all the students wrote the letter "x" to start the problem solving. It is interesting that no student wrote a different letter. Looking at grade 8 textbook Nhêze (1998) I observed that "x" is the predominant letter in word problem solving examples. Australian students used "x" but used other letters, too. Sometimes they used more than one letter. Some of the Australian students committed "reversal errors", for example two students used the letters a, b, c and formed the relationships a, $b+7$ and $c+13$ to indicate that a is the smallest quantity, b is larger by 7 and c is larger by 13 instead of a, $b=a+7$ and $c=a+13$. The "reversal errors" were not seen among Mozambican students as they used only one letter.

The major difficulty of the students categorized in "Partial Use" was related to the understanding of the relations between parts formulated in the Portuguese Language (PL): "...7 km farther than ..., 11 minutes longer than...". The PL is the language of instruction in all Mozambican schools and it is a second language for most students. During the interviews, I asked the students to read the problems and explain them using their own words. Often the students who produced incorrect equations interpreted the problems as "...the scout walked 7 km on Sunday,...Jorge spent 11 minutes washing the third car". The Australian students (several) understood the relations involved in the problems, so they were able to solve them by non–algebraic procedures. This difference can be understood as the Australian students had been presented the problems in their mother tongue.

The relationship "...7 km farther (longer)than ...formulated in PL is typical of mathematics classroom. It is not used much outside the classroom. Also it is difficult to translate it into the Mozambican Bantu Languages (BL). For example the statement "on Sunday they walked 7 km farther than they walked on Saturday":

Portuguese (PL): *"No Domingo eles andaram mais 7 km do que no Sábado".*

Macua (BL): *"Mucuaha yenthaya Dimingo, yahivicana mucuaha yenthaya Sábado i km 7".*

Gitonga (BL)*: "Ku Ndimingo abvô bva gimbindê gu bvinda ku li bvande na dzimbile dzi km 7 bva gimbila ku Sábado".*

The Portuguese statement can be translated as: On Sunday they walked more 7km than on Saturday. The word "mais" means "more" as well as the reading of the sign + . Therefore, for Mozambican students it is easy to relate the word "mais" to the sign +. This may be the reason why Mozambican students did not make "the concatenation for addition". For example, among Australian students the concatenation for addition was seen with four students, i.e., they wrote x7, x13, to mean 7 more than x and 13 more than x respectively. I don't know how far the English expression ...7 km farther (longer) than... tells Australian students that some quantity has to be added to another.

The Portuguese expression “mais...do que” equivalent to the English expression "farther... than" is used to compare two quantities and "do que" alone has no meaning. It presuppose the existence of other word before, e.g. "mais". However the students did not pay attention to whole expression. They only saw "mais" which means "more" and corresponds to the sign +. The students do not see the expression as a comparative one. This may be the reason why they wrote x+7+13=80 instead of x+x+7+x+13=80.

The expression can not be directly translated into Mozambican BL. For example, the Macua statement above can be translated as: the journey they walked on Sunday *exceeds* the journey they walked on Saturday by 7 km. The Gitonga statement can be translated as: on Sunday they walked. It *exceeds* in 7 km what they walked on Saturday. In both BL, the expression "mais...do que" is translated into expression of different meaning *"exceeds"* and a lot of description has to be added to make the relations clear.

3.3 The "No algebra" category

No student was categorized in "No algebra". However one girl (Julieta, 15) had first attempted to write an equation for problem 3, and as she did not succeed, decided to solve it verbally. In the following I present her explanation:

" I don't know the length of the first side, but I know that the second side is 3 cm greater than the first. The third is twice the first. Isn't it? This triangle is a square. It is a square. If I take away 3 from 63 I will have 60 and I can make a square. As the third side is twice the first, I make two equal sides. This is a square. Four equal sides. 60 divided by 4 is equal to 15. This is the solution. Now let me try the equation again".

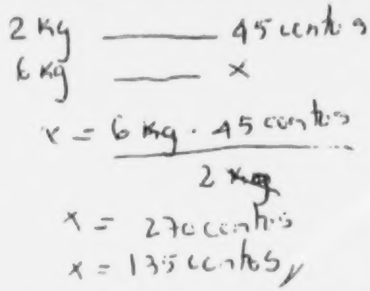
Julieta used the expression "the first side of the triangle" as the unknown and worked normally with it. Thus, using the relationships stated in the problem she organized the three different sides of the triangle into four equal sides that she named a "square". Each side of the "square" is an unknown and corresponds to "the first side of the triangle". The reasoning reminds us of the old Babylonian "algebra" as described by Radford (1996). The Babylonian algebra was characterized by the lack of symbolic representation because the alphabet had not actually been invented. So, words like length, breadth, area, were used in place of the unknown. Radford sees it as the beginning of the algebraic reasoning.

4. Three possible reasons why students resort to algebraic procedures

After observing that the students resort to equations to solve the problems, I interviewed secondary school teachers, to hear from them, what they think could be the reasons. I also studied school material looking for relevant tasks that can contribute to students resorting algebraic procedures. In that extra study I found some explanations that could be associated with:

4.1 The extensive use of "the rule of three algorithm"

The rule of three algorithm (crossed products, see Nádia in figure 4). This rule was extensively used during the colonial system of education in Mozambique. It was the only tool to solve specific problems. Despite of the new teaching materials to increase the diversity of techniques to solve problems of direct proportionality, many grade 7 teachers stress it in the classroom. It requires algebraic procedures. For the first time in grade 7, students write the letter x to represent the unknown quantity. I presented the following extra problem to 17 students of the study: *2 kg of fish cost 45 "contos". How much will 6 kg cost?* It is interesting to observe that the 17 students interviewed applied the rule as Nádia did below.



2 Kg —— 45 contos
6 Kg —— x

$$x = \frac{6\ \text{Kg} \cdot 45\ \text{contos}}{2\ \text{Kg}}$$

x = 270 contos
x = 135 contos

Figure 4: The rule of three algorithm (Nádia, 14)

How would Australian students solve this?. However, there are many ways to solve the problem without using any letter: take the price of 1 kg (22.5); make 6x22.5=135. Or let 6kg to be 3x2kg; now make 3x45=135. These arithmetic procedures were not seen among the 17 students.

4.2 The first word problems used in textbook to introduce algebraic procedures

The following problem is an from the grade 8 textbook used in the most Mozambican schools. *Rafael thought about a certain number. He added 3 and obtained 10. Which number he thought about? Let x be the number thought by Rafael. Thus, the equation of the problem is x +3=10, Nhêze (1998).*

In the example, there is no obvious need to let "x" be the number thought of by Rafael because the students "see" it (7). The problem, because of its simplicity, can make students think that, solving problems means writing equations and solving them .

4.3 The teachers attitude towards arithmetic procedures

Three teachers interviewed were unanimous in saying that, once the textbook deals with equations, the students should attempt to write them. During the tests, if the student did not show the equation, he could not get the full marks. These teachers' attitudes may contribute to students' devaluing of arithmetic procedures while equations are useful for complicated problems.

5. Conclusions

All the 43 Mozambican students interviewed attempted equations to solve the problems. 35% of 90 Australian students attempted some algebra. Maybe the Australian students have a much better basis of arithmetic than Mozambican students. In Mozambican schools memorizing is considered important and thus when students learn a procedure from the teacher follow it step by step even if they do not understand. Their reasoning is "the teacher does it and so do I".

The Mozambican students did not understand the relationships stated in the problems formulated in PL (linguistic problems). This is the reason why most student were categorized in "Partial Use". The Australian students, however, understood the relationships stated in the problems.

The reasons that lead Mozambican students to resort to equations can be associated with the extensive use of rule of three algorithm in grade 7, the first word problems (examples) used in textbooks to introduce algebraic procedures, and the teachers attitude toward arithmetic procedures during the learning of equations.

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PROTO-MATHEMATICAL CONCEPTS IN NORTHERN CHILEAN AYMARAS*

Vicente Neumann; Universidad Católica del Norte; Chile

Rafael Núñez; University of California at Berkeley, USA

Manuel Mamani; Universidad de Tarapacá, Chile

From the point of view of the cognitive linguistics, Lakoff and Núñez state that there are some concept schemes at the base of mathematical ideas. These schemes have the form of mathematical metaphors that are grounded in our body experience. In this research we find important differences between the traditional western mathematical metaphor such as described by la Lakoff and Núñez. We discus the implication of theses findings for the principles for an Intercultural Bilingual Education: Aymara-Spanish.

This work lies within the tradition of embodied cognition (For more details about this point of view, the reader can referred to the Gorge Lakkof proposal, an invited speaker of this PME24 conference). One of his claims of this point of view establishes that, because all human beings share certain features (i.e. neurological and anatomic characteristics, biped condition, frontal vision, etc.) there is a set of common corporal experiences that originates universal cognitive structures. These structures, that make sense to these basic embodied experiences, are mapped or projected on other conceptual domain, with its respective structural inferences (Lakoff and Núñez, 1997a). For example, there are receptors for movements, objects and locations in the organism, there are not receptcrs for time, so time can be understood in terms of objects, movements and locations in space. Therefore, cognitive structures directly related to the embodied experience, such as space, are mappings on other domains, such as time. Thus, in the previous example, "times" can be considered as "objects" and "movements" in space. This projection from one conceptual domain (the source domain) to another conceptual domain (the object domain) is called mapping.

Recently, Lakoff and Núñez described some interesting mapping in mathematics (1997a, 1997b). In this study we are concerned with (a) **ARITHMETIC IS OBJECT COLLECTION** -in this mapping numbers are objects of uniform length and the arithmetic operations are actions through which objects are collected-; (b) **ARITHMETIC IS OBJECT CONSTRUCTION** -where numbers are physical objects and arithmetical operations are activities in which these objects are constructed-; (c) **ARITHMETIC IS MOVEMENT**, thus the numbers are located in a path and the arithmetical operations are acts of movements through this path; and (d) **SETS ARE OBJECT CONTAINERS** -in this sense the categories are understood as regions (in the

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topographic sense) that contain a collection of objects. These containers, can also be considered as objects and can be included in other more inclusive containers.

Some authors mentioned that the Aymara language is remarkably different from the Indo European languages in concepts closely related to the field of mathematics, such as time, space, numbers and categories (i.e. Miracles and Yapita, 1975; Grebe, 1990). In spite of the anthropological statement, we sustain that Aymara uses the same grounding mathematical metaphor and that the reported differences may be within the linking metaphor level. The aim of this work is to study the universality of mathematical mapping proposed by Lakkof and Núñez, and try to extend its validity with a multidisciplinary approach that relates cognitive anthropology, intercultural psychology, and cognitive linguistics.

Method

In order to address the foregoing objective, we follow qualitative data analysis well established in linguistics. For further details, see the reference mentioned below in the procedures.

Subject

53 informants, men and women, between 35 and 80 years old, from Aymara communities of North Chilean cities (level of the sea altitude), valleys (between 1.000 and 2.000 meters of altitude) and village in the Andes plateau (more than 3.000 meters of altitude). Most of the subjects have incomplete elementary school; nevertheless, some informants are totally illiterate.

Instruments

Guideline of partially structured interview, with a section in Aymara and another one in Spanish; list of common expressions in Spanish and VHS camera.

Procedures

The subjects were interviewed in their homes and each session spent between 45 to 90 minutes In some cases more than one session was needed. We used the following analysis with the video recorder session: (a) Conceptual mapping method (Gibbs, 1990), (b) psycholinguistic analyses (Lakoff an Johnson, 1980), (c) conceptual networks (Turner y Fauconnier, 1995) and gestures analysis (McNaill, 1987, 1992).

Results

There were not people strictly mono linguist in Aymara. The habitants of the Chilean Andes plateau use Aymara as the natural language and, most of the oldest people, use a "Andean Spanish" to communicate with those who does not speak Aymara (Arnold and Yapita, 1996). As far as we now, this "Andean Spanish" is not just a dialect form of the Standard Spanish. It is seen rather as a different language with most words in common but a very different syntactic and semantic. One of the main aspects of this language is that it shares most of the inference and Aymara language postulate (about linguistics Aymara postulate, see Hardaman, 1988). An important number of the

Aymara valley people can understand but can not speak Aymara and they communicate in a Standard Spanish. In the cities, few Aymara people have ability for Aymara language, and, they use standard Spanish as natural language.

Because the fundamental metaphors are directly related with the anatomy and physiology of human beings, the authors postulate that the fundamental mathematical metaphors are universal as much as these body features are universal (Lakoff and Núñez, 1997a). In Aymara we found evidences for **ARITHMETIC IS OBJECT COLLECTION**, and for **ARITHMETIC IS OBJECT CONSTRUCTION**, but not for **ARITHMETIC IS MOVEMENT** nor **SETS ARE OBJECT CONTAINERS.**

The tendency of the informant to count spontaneously with the fingers, small stones or other objects provide evidence for the **ARITHMETIC IS OBJECT COLLECTION** metaphor. Moreover, during the arithmetical calculation, they use verbs such as *upjatña* (to have), *apaña* (to take) *apsuña* (to take from within) *apacaña* (to lower), and apxataña (to put on). For Aymara numbers are physical objects, that can be manipulated and Arithmetic is manipulation of physical objects. For example, the expression *kimsa apxatañani* (three plus), than is glossed in *kimsa* (three), *apxataña* (to put on), and *-ni* (with, possessive and enumerative suffix), illustrate that the three is understood as an object that is put upon others objects (others numbers). The expression *pusi kimsa paqallqu apsutawa*, (four [and] three are seven), is glossed in *pusi* (four), *kimsa* (three), *paqallqu* (seven), *apsuta-* (from *apsuña*, to take from within), and show that in the arithmetical operational numbers are objects that can be taken from within of something. In *p"isqat pay apaqatax kinsawa* (take two from five and we have three), is glossed in *p"isqat* (five), *pay[a]* (two), *apaqata-* (lowered, from *apacaña*, to lower), *x[a]* (on, over, from; location in space suffix), *kimsa* (three) -*wa* (emphasis suffix). In this sense, in arithmetic additions numbers are putting over, and in subtraction, numbers are lowered from some metaphorical place.

They do not have zero as a number; nevertheless, they use a related concept: *ch´usa* that means "nothing" or “empty”. Thus, the expression *P"isqa.t P"isq apaqata.x ch´usawa* (five less five is nothing left), is glossed in *p"isqa* (five), *-t[a]* (of), *p"isq[a]* (five), *apaqata* (lowered, from *apacaña*, to lower), *-x[a]* (on), *ch´usa* (nothing) *-wa* (emphasis suffix). On the other hand, we found the same construction in concrete object subtraction: *Kinsa mansana canasta.n utj.i.x Uka.t kinsa mansana apsut.ax, ch´usa.ki.wa* (There are three apples in the basket, then, we removed three apples, is absolutely nothing left). In these two examples, subtraction with abstract numbers and subtraction with apples, as in the western tradition, numbers operated in the same way that objects are manipulated.

The metaphor **ARITHMETIC IS OBJECT CONSTRUCTION** is supported mainly in Aymara numerical system by itself. Aymara numerical system is a decimal one, with twelve morphemes: nine unity roots and three relational roots. As we mentioned above, they do not have zero, although they have a concept to indicate absence of elements. The unity roots are *maya* (one), *paya* (two), *kimsa* (three), *pusi* (four), *pisq"a* (five), *suxta* (six), *paqallqu* (seven), *kimsaqallqu* (eight), and *llatunka* (nine).

The relational roots are *tunka* (ten), *pataka* (one hundred) and *waranka* (thousand). The rest of the numbers are made up combining these two types of morphemes (unity roots and relational roots). To do it so, they use the following suffix: *-ni* (possessive and enumerative suffix; can be translated as "with"), *-ta* (relational complement; noun suffix; can be translated as "of" or "from") and *-ru*.(location suffix; it mean "to", in the sense "to the village"). For instance, the morpheme by morpheme gloss of *tunkamayani* (eleven) is *tunka* (ten), *maya* (one), and *-ni* (with) -literally its mean *ten with one-*; *patak tunkani* (one hundred and ten) is glossed in *patak* (hundred) *tunka* (ten) *-ni* (with); its mean literally *hundred with ten*.

With the exception of expressions like "count from one up ten", there is not evidence of the **ARITHMETIC IS MOVEMENT** metaphor. For example, in Aymara it makes sense *payat tunkaru wakuña* (count from one up to ten), but not *tunkat mayaru wakuña* (count backwards from ten to one). Neither does it make sense to use sentences like *kimsat pisq"at pusi typinquiwa* (four is in the middle of three and five).This kind of metaphor requires the fiction motion that is unusual between Aymara people. Most of the western and Indo European cultures, can describe static things using a movement that really do not exist. Aymara people use the fiction motion in very restricted cases and, usually with borrowed Spanish, e.g. "the road goes through the village", "the drainage ditch passes near the house", etc. In the previous sentences, they use Spanish borrowed verbs as *pasaña* (go through, from de Spanish *pasar*). Moreover, when Aymara used fiction motion the cases are limited, because there are some kind of motion involved in the expressions, e.g., the people really walk on the way and water flows in the drain.

Up to our now, Aymara numbers have cardinality but nor ordinality. They can order and sequence; nevertheless, they do not use numbers for indicating the object location; instead of that, they indicate locations preceding or succeeding elements. Usually, they mentioned the first element as *nayrax* (ahead of) word that can be glossed in *nayra* (eyes) and *-x[a]* (on, place suffix). For the following they use *ukat q"iparu* (then) that can be glossed in *uka* (this), *-t[a]* (from, of), *q"ipa* (back, behind) and *-ru* (to). And the last element is mentioned as *q"ipax* (the last one) that came from *q"ipa* (back, behind) and *-x[a]* (on, over).

We did not find clear evidences for **SETS ARE OBJECT CONTAINERS** metaphor. Maybe, like others isolated rural communities, they made concrete object grouping but not abstract sets of mathematical elements. In many cases, the syntax allows to identify the group with the simple repetition of noun. For instance, *qala* (stone) and *qalaqala* (stone grouping), *cota* (lake) and *Cotacotani* is the name of a place where there are several little lakes very close to the others. Although there are Aymara lexicons for specific kinds of grouping such as *tama* (herd); *montuña* (group of objects, Spain borrowed) *juntuña* (grouping, borrowed from Spanish verb juntar, that means to join).

Therefore, we did not find scheme-containers. The different words and expressions mentioned before make reference to the group of individual objects, but not to a scheme-container, like a conceptual object that can contain elements. The abstract set idea does not appear in the Aymara language. The Aymara syntax does not permit expressing this idea. Aymara people can talk about to the set concept but with borrowed Spanish.

Discussion

The evidence of this research does not support ours main hypothesis. The differences between Aymara and Western mathematics are not only at the linking metaphor level but at the grounding metaphor level. Western and Aymara mathematics are similar in the ARITHMETIC IS OBJECT COLLECTION and ARITHMETIC IS OBJECT CONSTRUCTION metaphor, but not in ARITHMETIC IS MOVEMENT nor in SETS ARE OBJECT CONTAINERS. Moreover, western metaphor ARITHMETIC IS OBJECT COLLECTION and ARITHMETIC IS OBJECT CONSTRUCTION are examples of a more general metaphor NUMBERS ARE PHYSICAL OBJECTS and, as any objects, numbers are seen as being in the physical world with independence existence of other objects. In Aymara numbers are physical objects too, but not all the source domain inference (objects in space domain) are mapping to target domain (mathematics domain). Thus, in the natural Aymara language numbers can be manipulated, but they are not considered independent of the objects that are enumerated. The quantities are properties of the objects and do not have abstract independence existence that allow then to operate without reference to any particular object. At a first glance it seems like Aymara people do not have a high level of abstract thought. Besides, these same people have normal abstract level speaking in Spanish; they are able to make abstract mathematical relation as any other rural farmer.

The absence of ARITHMETIC IS MOVEMENT and SETS ARE OBJECT CONTAINERS metaphors in Aymara mathematics entails the absence of (a) zero as a number, (b) negative numbers and (c) subset of numbers. In ARITHMETIC IS OBJECT COLLECTION and in ARITHMETIC IS OBJECT CONSTRUCTION zero represents the absence of a collection or constructed objects, as the Aymara world *ch'usa* (empty, nothing) in both of these basics metaphors, zero is not the same kind of object as the other numbers. Only in ARITHMETIC IS MOVEMENT zero have the same mathematical properties; that is because in the latter metaphor zero is a location as any number. Besides, without mathematical fiction motion there are not opposite movement, which is the natural grounding for the negative numbers. In the latter metaphor, numbers are location on a path, zero is the starting point, the positive numbers are movement in one direction (the positive one) and the negative numbers are movement in opposite direction (the negative one). In addition, there can not be subsets of number, as the negative numbers or common denominator numbers. Although the mathematical Aymara system is more restricted in inference than the western system, it has the basis from which all the mathematical ideas come from. However, the mathematical Aymara system is broad enough for to make sense the Aymara mathematical experience of the village people in the Andes plateau (a good example of the procedures for distributing the community work or family heritage goods can be found in Neumann, Mamani & Núñez, 1999). Still, when the Aymara people of Northen Chilean cities and villages, or the Andes plateau interact with western people, use the traditional western mathematical system in the Spanish language. For instance, in some questions, certain Aymara informants make the calculation in Spanish and then translate the answer into Aymara. Therefore, when the situation or question are broader than of the Aymara daily experience, Aymara individuals are impelled to use traditional western mathematics in Spanish.

After the Pacific War on 1897, Chile annexed Aymara Peruvian territory and began a process named "*Chilenización*". The Chilenización was a State effort to introduce the Chilean national values, norms, legislation and culture onto the new territories. They impose the Spanish language onto Aymara peple, the private property system over the community Aymara property system, a new political organization that did not consider the Aymara political organization, etc. (Tudela, 1998). At this time, with the mass media and communication developing, the Aymara culture is going to become extinct in the Chilean territory in a few years. This catastrophic situation could be prevented with an intercultural bilingual education. Because cognitive linguistics is a broader approach –integrating cognitive psychology, anthropology and linguistics- this kind of analysis is very important for delineating the educational aims, methods and procedures. The intercultural bilingual education proposal can not be so simple as to teach Aymara and Spanish language, and the rest of the usual western educational subjects. The major Aymara conceptual schemes must be included in the curriculum. As in the Andean Spanish they had success in keeping the Aymara inferential system and the Aymara linguistic postulate onto some kind of Spanish language. The intercultural bilingual education must integrate the western knowledge tradition of preserving Aymara conceptual schemes. In the case of Aymara mathematics curriculum, as an example of the utility of this approach to the intercultural education, we can make some directions on the bases of these research results.

Teach mathematics in Aymara language. Therefore, we need to introduce new concepts into the Aymara language and thus, to introduce new conceptual schemas in Aymara thought.

Use extensively the grounding of a basic mathematical metaphor that already exists in the Aymara language. The Aymara has sufficient resources for copy with **ARITHMETIC IS OBJECT COLLECTION** and **ARITHMETIC IS OBJECT CONSTRUCTION** metaphor. Moreover, these mappings are nearest to the daily Aymara experiences.

To take into account Aymara Linguistic postulate in the interpretative translation of the western mathematical concepts onto Aymara language. "Linguistics postulates are the ideas or concepts or themes which permeate throughout and influence all aspects of a language" (Hardman, 1981, p. 11). The Aymara linguistic postulate are rather difference of the Indo Europeans languages and misunderstanding these postulates has remarkably consequences in the intercultural communication. Obviously, Aymara syntaxes, morphology, and grammar must also be considered.

Introduce Aymara writing language. Originally, Aymara is not a written language. However, reading and writing are some of the most important elements of the formal education. In fact, writing can not be avoided in formal mathematical teaching.

Extend the use of fiction motion to a broader common situation. This could be done through verbal mathematical problems. At the present days, Aymara expresses some ideas with fiction motion with Spanish borrowed. Thus, it could be easy to extend the use, with the same borrowed Spanish words to other common situation.

Introduce ARITHMETIC IS MOVEMENT *metaphor.* This metaphor needs a mathematical agents who is "A metaphorical idealized actor, that is, an idealized actor in the source domain of a metaphor characterizing some aspect of mathematics" (Lakoff & Núñez, 1997b, p.33). This agent can be a traveler, for instance, in the expression "1 is far away from 1,000,000". Aymara interpersonal relations are so important that they have a complex verbal inflection system where each tense involves interactions between two of the four Aymara grammar system persons. In addition they mark the difference between human and no human actions (In Aymara it does not exist "it" in the grammar system person) and, because kindness is very important for Aymara, the second person *juma* that mean "you and other belong to your group", is more important than the firsth person. The election of the grammar person for the mathematical agents in Aymara is not so simple and needs further research.

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THE CASE ANALYSIS OF SIX GRADERS' JOURNAL WRITINGS:
Using the "Framework for Analyzing the Quality of Transactional Writing"

Hiroyuki NINOMIYA
Hiroshima University Graduate School, Japan

The purpose of this paper is to establish and validate the "Framework for Analyzing the Quality of Transactional Writing" and to find the implications of using it to analyze 6th graders' journals. At first, from the literature review of preceding studies, the "Framework", which consists of "Holistic scoring", "Aspect of mathematics", "Levels of description", and "Explanation elaboration" is established. Then 6th graders' journal writings are analyzed using the framework established in this study. The analysis results in data shows evidence of themes within the journal writings. These themes, or characteristics, are discusses, as well as how the proper qualified transactional writings should be. Since both descriptive and normative domains are indicated, the validity of the framework is suggested.

1. Introduction

Writing activities in mathematics education has been the focus of much research. One of the research, Miller(1992), asserts that "*Writing is an active process that promotes students' procedural and conceptual understanding of mathematics.*" (p.354) Nakamura(1989) investigates the students' journal writings by means of both encouragement and assessment of their mathematical thinking. Ninomiya(1999) finds out the determiners of qualified transactional writings. Furthermore, Principles and Standards for School Mathematics (Standards 2000): Discussion Draft (NCTM, 1998) states that "*Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics*" (NCTM, 1998, p.94). Moreover, in some integrated math curricula, such as "The Connected Mathematics Project (CMP)", "The Interactive Mathematics Project (IMP)", which are based on NCTM's Standards, writing activities are used so frequently. For example, students are always asked to explain their ideas by written language in the curriculum of CMP(Lappan et.al., 1996, p.13), and students make "portfolios" at the end of each chapter in the curriculum of IMP(Green, 1997, p.34).

In this paper, Transactional Writing, defined by Ninomiya(1998(a)), is the focus, and determination of the quality of transactional writing is addressed. In order to assess the quality of students' writings, the "Framework for Analyzing the Quality of Transactional Writing", which consists of *holistic* and *analytic scoring*, is established from the literature review of preceding studies. Then, 6th graders' journals (about 12-hour lessons of "Solids") are examined using the framework to find out some of the characteristics of journal writing. Finally, with the findings from the analysis, the validity of the Framework is examined.

2. The Framework for Analyzing the Quality of Transactional Writing

Students' journals are roughly divided into two categories; *expressive writings* and *transactional writings* (Ninomiya, 1998(a), p.426). The former expresses the feeling or emotional aspects in their math study, whereas the latter explains the content of the mathematics that is learned in math classes. Although expressive writings are often used in the research of affection, the number of studies that show the relation between the students' emotions and their writing is very limited. However, the qualitative analysis of transactional writing has been discussed in several papers. Two of the leading papers that were written on the subject are by Van Dormolen(1985), and Shield & Galbraith(1998). Van Dormolen(1985) explains the framework of textual analysis, which has the three main concepts of *correctness of the content*, *global perspective*, and *adaptation to the students' abilities*. In his paper, notions which refer to linguistic representations, such as *kernels*, *aspects of mathematics*, and *levels of language*, are also presented. Based on Van Dormolen's study, Shield & Galbraith(1998) establish the coding system for the analysis of student expository writing, shown in Fig.1.

Explanation elaboration
Exemplar, Kernel, Goal statement, Justification, Link to prior knowledge or experience, Practice exercises
Aspects of mathematics
Theoretical, Algorithmic, Logical, Methodological, Conventional
Levels of language
Particular procedural Particular descriptive, Generalized procedural, Generalized descriptive,

Fig.1 Coding System of Transactional Writing
(Shield & Galbraith,1998)

Since some of students' typical activities are "write a letter to a friend" and "responding to a student's difficulty", the statements of *practice exercises* which is in the category of *explanation elaboration*, play a certain role in the students' writing activities. In fact, Shepard(1993) also points out the importance of creating *practice exercises* (=problem posing) in writing activities(Shepard, 1993, p.291) ; however, when journals are written for the students' own sake, *practice exercises* are not expected to be needed, and such statements ought to be included into one of *exemplars*. Despite being beneficial research, Shield & Galbraith neglect the importance of "metacogintion". One of the important roles of writing activities in mathematics education is "expressing metacognition" (Kameoka, 1996, Shigematsu et.al., 1998). Since metacognition strongly influences mathematics learning (Shigematsu, 1990, p.76), the statements about metacognition are very effective for letting students express their own ideas or thinking. It is also reported that the "statements of metacognition" play important roles such as "control the exemplars", "explaining the exemplars" and so forth (Ninomiya, 1999). Therefore, "*statements of metacognition*" must be included as one of the elements of *explanation elaboration* in Fig. 1.

Aspects of mathematics can be understood in the following ways. The first aspect discriminates between writings that is either conceptual or procedural. *Theoretical* and *logical* are more or less conceptual, whereas *algorithmic* and *methodological* are procedural. The second aspect looks at whether the writing is either individual or relational. *Theoretical* and *algorithmic* are more or less individual, whereas *logical* and *methodological* are relational. On the other hand, *conventional* is explained as "conventions, how to name a diagram, and write a proof." (Shield & Galbraith, 1998, p.33) This will be another category of *aspects of mathematics* in this paper.

Since the coding system by Shield & Galbraith(1998) assess only the quality but do not assess the levels of writings, the holistic scoring, which is mentioned in

Ninomiya(1998(b)), is also used in the case analysis of this paper. Fig.2 summarizes the above discussion: the "Framework for Analyzing the Quality of Transactional Writing".

Holistic Scoring: Using the "Rubric" for each assignment

Analytic Assessments
- **Aspects of mathematics**
  - Conceptual - individual, Procedural - individual
  - Conceptual - relational, Procedural - relational, Conventional
- **Levels of language**
  - Particular - procedural, Particular - descriptive,
  - Generalized - procedural, Generalized - descriptive,
- **Explanation elaboration**
  - Goal statement, Kernel, Exemplar, Statement of metacognition,
  - Justification, Link to prior knowledge or experience,

Fig. 2 Framework for Analyzing the Quality of Transactional Writing

3. Procedure of the Case Analysis of Six Graders' Journals

Journal writing has been an activity for 35 six-graders (16 boys and 19 girls) in a public elementary school in Hiroshima, Japan, since Nov. 1999. They are in the same class and have the same mathematics lessons. After every math class, they write in their journals. Since they started their journal activities only a few months ago, they are still learning how to do this activity properly and still require the teacher's guidance. In this paper, their very first 12 hours of journal entries are examined. The topics of lessons are categorized into three groups as follows.

Group A: lessons about concepts (1st, 2nd, 3rd, and 4th)
Group B: lessons about procedures (5th, 6th, 7th, 8th, 10th, and 12th)
Group C: practice exercises (9th and 11th)

The students' journals are analyzed by means of the "Framework for Analyzing the Quality of Transactional Writing", or by the following two ways of *holistic scoring* and *analytic assessments*. The rubric shown in Fig. 3 is used for the holistic scorings of these 12-hour-math-class journals.

5 points: showing the main idea of the class clearly with kernels or other important ideas
4 points: showing the main idea of the class with some appropriate exemplars
3 points: showing the main idea of the class
2 points: showing some concrete statements, but without reference to the main idea of the class
1 points: showing some statements about math study, but not concrete enough
0 points: showing just feelings or learning impressions

Fig. 3 The Rubric for the Holistic Scoring

Each journal is also examined using the *analytic assessments*, which looks at *aspects of mathematics*, *levels of language*, and *explanation elaboration*. Since the students have had very limited experience with journals, they tend to write statements which are not concrete enough, or something emotional. Although expressive writing is not the main point of this case analysis, two more categories in *explanation elaboration*, "the statement which is not concrete enough" and "learning impression", are added because of the number of such "non-transactional" statements.

Scoring the data of *analytic assessments*, the following procedures were adopted. In order to know the tendency of each category, subordinate points were defined. Since the *conceptual - individual* and *conceptual - relational* are both "conceptual", the subordinate point of conceptual was counted as the total of both frequencies of occurrence. The subordinate points of *procedural(A)*, *individual*, *relational*, *particular*, *generalized*, *procedural(L)*, and *descriptive* are likewise. The score of *conventional* was to be the frequency of occurrence. Also, the score of each categories in *explanation elaboration* was the frequency of each occurrence.

Moreover, for more detailed analysis, students were divided into three groups according to the standard score (z-score) of their journals' holistic scores, which is as follows.

Above Average (upper) : $z>0.5$, $n=11$
Intermediate (middle) : $-0.5<z<0.5$, $n=13$
Below Average (lower) : $z<-0.5$, $n=11$

Besides the data of the students' journals, the results of the "Affective Characteristics Test (Ito, 1995)", the "Instrument for Measuring Metacognitive Ability (Shimizu, 1995)", and the data of the students' achievement, are also used in this analysis.

4. Results and Discussion

4.1 The Analysis of Holistic Scores

The number of students of each holistic score by each lessons is shown in table 1.

Table 1 The Total Number of Students in Each Holistic Score

	Group A				Group B						Group C	
	1st	2nd	3rd	4th	5th	6th	7th	8th	10th	12th	9th	11th
5 points	0	0	0	0	1	0	1	0	1	1	0	1
4 points	1	1	1	4	5	3	2	7	4	4	0	3
3 points	5	7	3	11	11	12	18	7	18	19	3	12
2 points	10	19	16	11	17	15	6	15	5	3	4	2
1 points	10	7	9	8	0	2	7	4	4	4	18	14
0 points	9	1	4	1	1	1	0	1	2	3	9	1
average	1.40	2.00	1.64	2.26	2.63	2.42	2.53	2.44	2.62	2.59	1.03	2.15

The line labeled "average" is the average of all students' holistic scores in every class period. The average of all in group A is 1.82, Group B : 2.54, and Group C : 1.59. It is obvious that the average of Group B is higher than that of Group A. In fact, the result of a t-test indicates a significance of below 1 %. Besides, the preceding study shows that the average holistic score of sixth graders, who have had 2-years experience in journal activities, is 3.25 with the same 5-point-scale rubric (Ninomiya, 1999), and there is a significance of below 1 % with the data of Group B. Considering that this is the very beginning of their journal writing activities and the class teacher has tried to teach them how to write, it is natural to think that the students in Group B are still under achieving in their ability to write. The data in Table 1 shows the development of their writing abilities.

Yet, there still is the possibility of the influence from the difference of mathematical contents in each classes. Although more data is needed for the strict discussion, some implications are indicated in the following part of this paper.

4.2 The Analysis of the Correlations with Holistic Scores

Here is another table (table 2) which shows correlations between holistic scores of journals with students' mathematical abilities.

Table 2 The Correlations with Holistic Scores

	Holistic Scores of Journals		
	Group A	Group B	Total
Affective Characteristics (General)	0.22	0.30	0.35*
Affective Characteristics (Self Evaluation)	0.32	0.43*	0.46**
Metacognitive Ability	0.35*	0.34*	0.40*
Achievement : Mathematical Thinking (general)	0.40*	0.48**	0.51**
Mathematical Thinking (content)	0.26	0.34*	0.40*
Representation / Processing (general)	0.17	0.27	0.30
Representation / Processing (content)	0.40*	0.45**	0.53**
Knowledge / Understanding (general)	0.37*	0.44**	0.51**
Knowledge / Understanding (content)	0.07	0.18	0.21

** : $p<0.01$, * : $p<0.05$

Table 2 indicates that the students who get high score in either affective characteristics or metacognitive abilities tend to write "good" journals. Those who have a high *self-evaluation*, or self confidence, tend to have this characteristic especially. There also are rather strong correlations between achievements and journals. General achievement of *mathematical thinking* or *knowledge / understanding* affects a lot of their journals, whereas the correlation between *representation / processing* in the content (in this case, the representation or processing of "solids") with journals is strong. From the comparison of the data of Group A and Group B, the correlation with Group B is much stronger, which means that the more students have had the experience of journal activities, the more their journals correspond to their mathematical abilities.

4.3 The Analysis of the Correlations with Analytic Assessments

The correlations of the students' journals with the data of Analytic Assessments is shown in table 3.

4.3.1 Some Findings with *Aspects of Mathematics*

Table 3 indicates that the correlations of *conceptual*($r=0.62$), *procedural(A)* ($r=0.54$), and *individual*($r=0.86$) in Group A are strong, whereas those of *procedural(A)*($r=0.36$) and *relational*($r=0.42$) in Group B are strong. However, the detailed correlation of journals with *conceptual* and *procedural(A)* in Group A are totally different. Although the correlation of *conceptual* is stronger in upper and middle students, the correlation of *procedural(A)* is stronger in lower students. The former means that those journals which indicate the conception of mathematics tend to be "good", but the latter shows a different situation; the existence of students who have got the really low *procedural(A)* point (=those who can not write anything). In short, the tendency of students' journals in Group A is *conceptual and individual* whereas those of Group B is *procedural and relational.*

Table 3 The Correlations with Analytic Assessments

		Holistic Scores of Journals									Self-Evaluation	Meta. cognition
		Group A				Group B				Total		
		Upper	Middle	Lower	Total	Upper	Middle	Lower	Total			
Aspects of mathematic	**Conceptual**	0.67•	0.78••	0.20	0.62••	-0.31	-0.09	0.57	0.05	0.32	0.23	0.31
	Procedural(A)	-0.10	0.12	0.81••	0.54••	0.26	-0.02	0.02	0.36•	0.61••	0.16	0.15
	Individual	0.77••	0.85••	0.73•	0.86••	-0.12	0.55	0.44	0.02	0.50••	0.06	0.08
	Relational	N/A	N/A	N/A	N/A	-0.01	-0.50	0.34	0.42•	0.51••	0.32	0.36•
Levels of language	**Generalized**	N/A	N/A	N/A	N/A	-0.29	-0.26	0.34	0.48••	0.51••	0.35•	0.37•
	Particular	0.77••	0.85••	0.73•	0.86••	0.40	0.04	0.26	-0.21	0.51••	0.05	0.08
	Procedural(L)	0.19	0.28	0.83••	0.74••	-0.25	0.22	0.25	-0.17	0.75••	0.27	0.26
	Descriptive	0.51	0.74••	0.40	0.36•	0.44	-0.36	0.40	0.37•	0.13	0.07	0.20
Explanation elaboration	**Goal Statement**	0.60	N/A	0.30	0.46••	0.03	-0.19	0.34	0.01	0.19	0.29	0.09
	Kernel	N/A	N/A	N/A	N/A	0.10	N/A	N/A	0.20	0.20	0.26	0.33
	Exemplar	0.64•	0.84••	0.79••	0.86••	0.52	0.53	0.84••	0.76••	0.83••	0.31	0.38•
	St. of Metacog.	0.35	0.47	-0.07	0.62••	-0.26	-0.21	0.02	0.22	0.63••	0.38•	0.50••
	Justification	0.38	N/A	N/A	0.33	N/A	N/A	N/A	N/A	0.22	0.23	0.26
	Link to Pr. K./E.	-0.13	0.23	N/A	0.14	-0.11	0.12	N/A	0.20	0.38•	0.30	0.40•
	Not Concrete Sta.	-0.35	-0.30	0.39	-0.46••	-0.13	-0.13	-0.22	-0.26	-0.42•	-0.51••	-0.23
	Lear. Impression	-0.22	-0.15	-0.01	-0.22	0.19	-0.03	-0.49	-0.20	-0.21	-0.09	-0.35•

＊＊ : $p<0.01$, ＊ : $p<0.05$

The change from *conceptual* to *procedural* is due to the difference of the topics in their classes; as Group A is the lessons about the concepts, whereas Group B is about the procedures. On the other hand, the change from *individual* to *relational* is due to the progresses of their journals. Since the students have become to be able to write more sophisticated journals, they can also write some relational statements.

4.3.2 Some Findings with *Levels of Language*

The correlation with *generalized* increases a lot whereas the one with *particular* decreases between Group A and B. This may indicate the improvement of the quality of their journals, as they have become to be able to write some generalized statements. On the other hand, the correlation with *procedural(L)* decreases. Through the experiences of their journal activities, they have become to change to "not to write just the processes of what to do".

4.3.3 Some Findings with *Explanation Elaboration*

The correlation with *exemplar* is very strong, and has a little differences between Group A and B. This means the existence of *exemplars* is critical in any situation for transactional journals. Comparé to Group A and B, the correlation with the *statements of metacognition* have become weak but the correlation of *kernels* or *the statements of the links to prior knowledge or experience* have become exist. It seems that some of the former statements have become to the latter ones.

Further more, those journals which mainly state either *not concrete statements* or *learning impressions* are considered to be low levels; however, the negative correlation with *not concrete statements* in Group B is rather weak. This may indicate that some students who write *not concrete statements* in Group B still be a good writer as they may write some more concrete statements. On the other hand, correlations of *learning impressions* in Group A and B are almost the same, even though the students have had the instruction of how to write. The existence of *learning impressions* itself seems not to affect to the quality of transactional writings.

4.3.4 Some Findings with Self-Evaluation or Metacognition

Since the correlation between the abilities of self-evaluation and metacognitive abilities is rather strong ($r=0.43$, $p<0.01$), the correlations of journals to these two seems similar. Those who have high self-evaluation and metacognitive ability tend to write more *relational* and *generalized* journals. Their journals also include a lot of *exemplars*, *statements of metacognition*, and *links to prior knowledge or experience*. However, while there is a strong negative correlation between the number of *not concrete statements* and self-evaluation ($r=-0.51$), the negative correlation with metacognitive ability is rather weaker ($r=-0.23$). Moreover, there is quite strong negative correlation between the number of *learning impression* and metacognitive ability ($r=-0.35$), whereas there is almost no relation with self-evaluation ($r=-0.09$). That means the students who have high self-evaluation tend strongly to avoid the *not concrete statements*, but don't care if their journal include the *learning impressions*. On the other hand, the students who have high metacogintive abilities tend strongly to avoid the *learning impressions* and to avoid *not concrete statements* in some level.

4.4 Discussion

4.41 Implications from the 6th graders' journal writing

In the situation of the very first stage of their activities, the characteristics of 6th graders' journal writings are found as follows.

(1) Affective characteristics, matacognitive abilities, and the achievement of mathematics strongly affect to the quality of transactional writings.

(2) Students tend to write rather *individual, particular*, and *procedural* journals at the beginning, while they improve their journals to *relational* and *generalized* ones through the experiences of journal activities with the proper instructions.

(3) The contents of the math classes affect to the *aspects of mathematics*.

(4) Qualified journals tend not to include the statements either *not concrete statements* nor *learning impressions*.

(5) Those who have high self-evaluation and metacognitive ability tend to write more *relational* and *generalized* journals, and their journals include a lot of *exemplars*, *statements of metacognition*, and *links to prior knowledge or experience*.

4.42 Discussion on the validity of the Framework

Now, we try to examine the validity of the "Framework for Analyzing the Quality of Transactional Writing". The proper framework should have both descriptive and normative domains. The former should indicate the present characteristics, its improvements, affection from other factors, and so forth, while the latter tells us how the proper qualified transactional writings should be. The discussion in this paper tells us the characteristics of the very first stage of 6th graders' journal writings. Improvement of their journals, and the correlations with other abilities are also indicated. Moreover, the typical situations of the proper qualified journals are also found out. Thus, it is acceptable to say that the "Framework for Analyzing the Quality of Transactional Writing" has been validated so far at this point.

5. Concluding Remarks

In this paper, the 6th graders' journal writings are examined by the "Framework for Analyzing the Quality of Transactional Writing". First of all, the framework is established with several precede papers. Then, 6th graders' journals about 12-hour lessons of "Solids" are examined with the framework. Some of the characteristics of 6th graders' journal writings are found out, as well as how the proper qualified transactional writings should be. Since both descriptive and normative domains are indicated, the validity of the framework is suggested.

Since this case analysis is just about the very beginning of their activities, more analysis is needed to reveal the whole situation of 6th graders' journals, as well as to validate the "Framework for Analyzing the Quality of Transactional Writing".

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LISTEN TO THE GRAPH:
Children's matching of melodies with their visual representations

Steven Nisbet & John Bain
Griffith University, Brisbane, Australia

Abstract

This paper is a report of matching of melodies with line graphs and music notation by children (aged 10 & 11 years). The study found that children are able to match the up-down contour of melodies with their visual representations, and that matching-task performance is positively related to mathematical ability in the case of line graphs, and musical ability in the case of music notation. The results suggest that visually impaired students and students with a preference for aural learning may be able to perceive the shape of graphs through auditory output of graphical calculators or computers. It was found also that the children used global processing more readily than local analytical processes and were able to detect global features such as overall shape more easily than local details such as interval sizes.

Introduction

This paper is a brief report of a major study of children's ability to match the up-down contour of melodies with the up-down contour of two visual representations, namely line graphs and music notation (Nisbet, 1998). The importance of this research for mathematics education lies firstly in the notion that, if students can perceive up-down contour through auditory perception, then those who are visually impaired, or those students who have a preference for learning in the aural modality, may be able to perceive the shape of a graph through its auditory representation rather than the usual visual representation. Hence these students may gain a richer understanding of the mathematical function normally represented visually by the graph. Secondly, studies in this area may provide some clues to an explanation of findings by researchers (e.g. Geohegan, 1991; Gregory, 1988; Rauscher, Shaw & Ky, 1993; Shaw, 1997) that experience with music assists children to develop mathematical skills, particularly spatial task performance.

Connections between mathematics and music have been of interest to some mathematicians, scientists and musicians for many years dating back to Pythagoras, and there are several reports in the literature of the mathematical foundations of music (e.g. Bronowski, 1976; Nisbet, 1991; Thomsen, 1980). However, research into the matching of melodies with their visual representations has received only limited attention in the fields of music perception and experimental psychology, and even less in the field of mathematics education. Nevertheless some relevant studies from the areas of music perception and experimental psychology provided a stepping-off point for this study.

Firstly, Morrongiello & Roes (1990) found that children's matching of melodies with line graphs is influenced by auditory factors (Western-style tonal melodies were easier to match with graphs than atonal melodies), contour complexity (more changes of up-down direction made matching more difficult), and extent of the children's musical training (children with musical training were more successful with tonal melodies, but not with atonal melodies). Secondly, Balch & Muscatelli (1986) established that the matching of melodies with visual contour markers by young adults is influenced by modality condition (tasks in which the visual stimuli were presented first were performed better than those in which the auditory stimuli were presented first), presentation rate (i.e. matching performance improved as the speed of notes decreased from 5 notes per second to 0.5 notes per second), contour complexity (how many times the up-down contour changed), and musical training (whether or not the subjects learned a musical instrument). Thirdly, in a review of sensory modality studies, Friedes (1974) came to the general (but not universal) conclusion that intramodal tasks (e.g. judging whether two melodies match) are easier than cross-modal tasks (e.g. judging whether the up-down contour of a melody matches that of a graph).

Fourthly, Das, Kirby & Jarman (1979) have refined the Luria model of cognitive processing (Luria, 1973) which describes processing of visual and auditory stimuli in terms of simultaneous processing (i.e. many elements and their inter-relationships at one time) and successive processing (i.e. elements of a sequence one at a time). Fifthly, studies by Navon (1977, 1981) have determined that in visual perception, global processing takes precedence over local analytical processing. In other words, global shape is perceived before local details. Similar results have been found for melodic recognition by Dowling (1982) and Trainor & Trehub (1993). However, by contrast, Schwarzer (1997) claims that analytical processing of melodies takes precedence over holistic processing. Also related to this study is the work of Palmer (1990) who found that task instructions given to subjects positively influence their attention to specific features of visual stimuli. Perhaps instructions to take special note of local details would improve analytical performance. Similarly, instructions to take special note of global shape would improve holistic performance.

In music education circles it has been claimed that music education should develop students' ability to integrate information gained from auditory and visual perception (Walker, 1992). Such an integration is not normally required in mathematics education, however, if the up-down contour of a graph can be represented auditorally and if students are able to perceive its shape through the auditory representation, then this may possibly be a method to assist visually impaired students and students with a preference for aural learning perceive and understand the shape of graphs.

The study

This study was conducted in three stages and addressed the following specific research questions:

- Do visual factors influence the matching of melodies with line graphs to the same extent as auditory factors?
- What is the role of mathematical ability and experience in the matching process?
- What is the role of sensory modality in the matching process?
- What is the role of cognitive processing ability in the matching process?
- Do children use global processing more than local processing in the matching process?

- Do instructions to attend to local and global features improve matching-task performance?

Participants

The participants in the experiments of the study were children (aged 10 to 11) in Grade 5 and 6 classes at two government primary schools in suburban Brisbane.

Materials

The auditory materials used for the study were recordings of nine-note melodies played on an electronic keyboard using a familiar piano sound. The visual materials were line graphs and samples of music notation - some which matched the up-down contour of the recorded melodies and others which did not match. To investigate the effect of visual factors in the matching process, two formats were used for the graphs and music notation: (i) conventional format which had melodic pitch represented on the vertical axis (low pitch below high pitch) and time shown on the horizontal axis from left to right, and (ii) non-conventional format which had pitch on the horizontal axis (low pitch to the right of high pitch) and time on the vertical axis (from top to bottom). The non-conventional music notation also used different note symbols and staves to the conventional notation. See Figures 1 and 2.

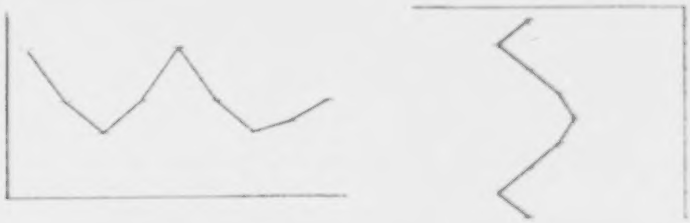

Figure 1: Conventional (left) and non-conventional (right) line graphs

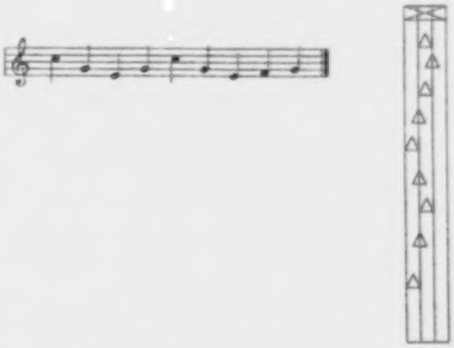

Figure 2: Conventional (left) and non-conventional (right) music notation

Experiment 1: Matching melodies with line graphs

The purpose of this experiment was to determine (a) whether visual factors influence the matching of melodies with line graphs to the same extent as auditory factors as demonstrated by Morrongiello and Roes (1990), (b) the role of mathematical ability and experience in the matching process, and (c) the effect of sensory modality in the matching process. Children in Grade 5 were asked to match melodies and line graphs presented in four sensory-modality conditions – melody to graph, graph to melody, graph to graph and melody to melody. ('Melody to graph' means that the melody was presented to the child first followed by the graph.) Half of the melody/graph pairs matched, and half did not. The procedure was that a pair of stimuli were presented to the child, and then the child was asked if the two matched or not, and the child's response subsequently recorded.

Manipulation of the visual format (conventional versus non-conventional graphs) and contour complexity showed that the matching process was influenced by visual/graphical factors as well as by auditory/melodic factors. Matching of melodies with conventional format graphs was superior to matching with non-conventional graphs. It was also found that intramodal tasks were superior to cross-modal tasks, in accord with much of the sensory-modality literature (Friedes, 1974) rather than with the *contour abstraction hypothesis* of Balch and Muscatelli (1986), which proposed that visual-first items are performed better than auditory-first items. However, within the intramodal and cross-modal categories, visual-first tasks were superior to auditory-first tasks. As a result of this, it was proposed that the matching process involved a comparison of an abstraction of the first-presented stimulus and the second stimulus, with the need for recoding from one modality to the other in the case of cross-modal tasks.

A positive effect of mathematical ability was revealed in this first experience, and evidence relating to type of visual format pointed towards the effect being attributable to mathematics experience (e.g. experience with graphs), rather than just mathematical ability. A close relationship between mathematics experience and mathematical ability was noted. Mathematics experience is a component of the mathematics ability measure used, and hence the role of mathematics experience is implied in the effects of mathematics ability. Positive effects of musical ability and musical training were observed also (as found in the study by Balch & Muscatelli, 1986) but the effects were limited to high-complexity visual-to-melody tasks (similar in part to the analogous mnelodic conditions in the study by Morrongiello & Roes, 1990). It was noted that as with mathematics ability and experience, a close relationship existed between these two factors, music experience and musical ability. Music experience is a component of the musical ability measure, and hence the role of music experience is implied in the effects of music ability. Despite the existence of these ability/experience factors in the matching process, they were overshadowed by the effects of sensory-modality condition and contour complexity.

Experiment 2: Matching melodies with music notation

The effect of visual factors on the matching process was further investigated in Experiment 2 with the use of music notation as the basis of the visual materials. Again, it was demonstrated, through the manipulation of notation format and contour complexity, that the matching process was influenced by visual as well as auditory factors. Also, the sensory-modality effects noted in the first experiment were observed again (intramodal tasks were performed better than cross-modal tasks), although performance levels indicated that matching melodies with music notation was more difficult for the children than with line graphs. Results for the visual-to-melody condition demonstrated children's greater difficulty of abstracting contour from music notation (compared to line graphs) and confirmed previous claims that the process of reading music is more complex than the cross-modal transfer of auditory and visual information.

Musical ability and music experience were positive factors only in the melody-to-visual condition, not surprising for that condition given that music students would be familiar with the task of listening to a melody and studying the music notation. The children would be no more familiar with that than reading music notation and performing the notes, so it appears that the difference in music ability/experience effects between notation-to-melody and melody-to-notation conditions may be explained by the difficulty the children (even the musically able) have with abstracting contour from notation compared with a likely advantage musically able children have with abstracting contour from melodies.

Overall, the effects of ability factors were again overshadowed by the effects of modality condition and contour complexity. The fact that musically experienced children did not significantly outperform their inexperienced counterparts overall suggests that, generally speaking, children who have been studying a musical instrument for one or two years find reading music notation a difficult task despite the fact that they learn music.

Experiment 3: The role of cognitive processing

The issue of abilities was continued in Experiment 3 which examined the matching of melodies and their visual representations this time with respect to abilities in simultaneous and successive cognitive processing, based on the Luria model of cognitive processing (Das, Kirby & Jarman, 1979; Naglieri & Das, 1990). This model was shown to be a basis for understanding children's abilities in the matching process, but the relationships between sensory-modality condition and simultaneous and successive processing were not as well defined as originally proposed. Simultaneous cognitive processing was a significant positive factor in the performance of tasks in the two visual-first modality conditions (visual to visual and visual to melody) and the low-complexity melody-to-melody condition, whereas successive cognitive processing was a significant positive factor in all four sensory-modality conditions.

The results indicate that simultaneous processing was involved not only with the inter-relating of features of visual stimuli but also with the "chunking" of melodic phrases (in the case of low complexity examples). The effect of successive cognitive processing ability was attributed firstly to the processing of the sequence of notes of a melody, and secondly to the consecutive presentation of the two stimuli. The results show that the cognitive processing required for matching tasks can be determined by considering the modality of the first-presented stimulus, and the consecutive nature of the presentation of pairs of stimuli. This conclusion also was consistent with the matching-process model arising from Experiments 1 and 2, which proposed that matching entailed abstraction of the contour of the first-presented stimulus and comparison with the second stimulus. Experiment 3 also confirmed the assertion made in consideration of the results of Experiments 1 and 2, that music notation is more complex visually than line graphs, and thus requires a higher level of simultaneous cognitive processing to abstract the contour given the complications of the perceptual and symbolic features.

Experiment 4: Recognising local and global features during the matching process

The local and global features of the melodic and visual materials and their associated processing strategies were the major issues investigated in this experiment and the following one. Children's recognition of differences in the materials at the local and global levels was examined with respect to analytical and global processing, and presentation rate. Experiment 4 provided limited evidence for the global precedence hypothesis (Navon, 1977, 1981). Under both modes of instruction global changes were distinguished from local changes, and were detected at faster presentation rates. Moreover, children appeared to have difficulty in recognising interval changes which don't violate the overall up-down contour. The results showed that children were more sensitive to changes in line-graph items compared to music-notation items, and that processing instructions did have some effect on the children's responses. Limitations in the methodology restricted the extent to which the results could be interpreted in terms of global precedence and the effect of instructions. The next experiment sought to remedy these limitations.

Experiment 5: Global or local processing?

Subsequent to the noting of the limitations to Experiment 4, the methodology was refined in terms of materials, tasks and methods of response. The children were asked to indicate whether pairs of melodies and graphs presented concurrently were (i) identical, (ii) had different overall shape, or (iii) had the same overall shape but had changes to interval sizes. Two types of instructions – global and local - were used to direct the children's attention to the global or local features, and only two presentation rates were employed – fast (4 notes per second) and slow (1 note per second).

It was found that global processing took precedence in the matching of melodies and line graphs, confirming Navon's (1971, 1981) global precedence hypothesis. Global information with respect to overall contour was accessed more easily and more quickly than local information in the form of interval sizes. This is consistent with results from studies in melodic perception which have shown that children prefer to use global attributes such as contour rather than local attributes such as interval sizes (Dowling, 1982; Trainor & Trehub, 1993). Attention to local and global properties was able to be manipulated by mode of instruction (as predicted by the work on the role of attention by Palmer, 1990), but only at the faster presentation rate. The recognition of global changes (changes to overall contour) was reinforced by instructions to act holistically and hindered by instructions to act analytically, and the recognition of local changes (changes to interval sizes) was reinforced by instructions to act analytically, and hindered by instructions to act holistically.

Decreasing the presentation rate appeared to lead to a loss of cohesion of local and global melodic information in terms of the children's perception of relative interval sizes. Although the children recognised global-change items reasonably well, they incorrectly reported more differences for global-change items compared to local-change items. The children perceived differences between the two types of items but interpreted them quantitatively as well as qualitatively. Overall, it was shown that, in the matching of short melodies with line graphs, global information is accessed more easily and more quickly than local information, and that children's matching strategies can be manipulated through instructions.

Conclusions and implications

The main conclusion from this study is that children are able to *listen to a graph* and detect the overall shape of the graph. The up-down contour of a graph can be abstracted from visual and auditory stimuli. This suggests the possibility of using auditory representations of graphs to assist visually impaired students and students with a preference for aural learning perceive the shape of a graph and comprehend the mathematical function underlying it. With the advances in mathematical technology (e.g. graphical calculators and computer software) it seems feasible to produce auditory as well as visual output from a mathematical function. Further research is required into how this technology may assist visually impaired mathematics students in understanding functions in algebra, calculus and statistics.

Other conclusions from the study include the following:

- Children's performance at matching melodies and their visual representations is influenced by visual as well as auditory factors.
- Reading music notation is a difficult task for children, even after two or three years' musical tuition. Music notation contains much abstract as well as perceptual information. This information is contained in the local detail of the melodies and their visual representations, and it is not as easily detected as the global shape.
- Experience with the respective visual materials assists in the matching-task performance – mathematics experience for matching with line graphs and musical experience for matching with music notation.
- In the matching tasks with line graphs, children use global processes more readily than analytical processes, and perceive global features more easily. This result corresponds with the claim of the van Hieles (1958) that, in terms of visual stimuli, children first learn to recognise overall geometric shape (Level 1), and at a later stage are able to analyse specific local properties (Level 2).

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TEACHING MATHEMATICS VIA INTERNET: WRITTEN INTERACTIONS BETWEEN TUTOR AND STUDENT

Lourdes Figueiras Ocaña
Departament de Didàctica de les Matemàtiques i de les Ciències Experimentals
Universitat Autònoma de Barcelona
Spain

Abstract: Our aim in this research is to explore the communication process between a tutor and a single student participating in a tutorial imparted at a distance via Internet on the solution of systems of linear equations. Among the diversity and richness of the new situations arising from this context, we concentrate on their written interactions via e-mail and develop for its analysis a methodology based on discourse analysis.

We find that the sequence "Question-Response-Evaluation" considered to be the simplest of student-teacher interactions has to be enlarged by more complexe interactions taking the form of nested loops including incition to self supervision and emotional reactions. These loops on the one hand increase the participation of the student, but on the other usually end abruptly due to the special restrictions imposed by the situation of learning at a distance.

Introduction

In this paper, we will present how the case of an 18 year-old high performance athlete studying mathematics –solution of systems of linear equations- has been carried out. Due to sport competitions, he spends long training periods in different places and thus loses the possibility of assisting classes regularly. The student is then tutorized by the researcher and maintains no contact with his current mathematics teacher or colleagues.

As a theoretical base of our investigations, we agree that personal learning of mathematics is inseparable from the social practices which severely constrain the manner and quality in which learning takes place (Crook, 94; Brown, 94). We emphasize the need to consider the projection to secondary Mathematics education of the current changes in adult education paradigms presupposing an inherent capacity for autonomous learning (Albero, 99). This change goes hand in hand with an increasing use of multimedia learning environments in schools. It is necessary to reflect on their impact on the students' opportunities to interact socially both at school and at home.

We must also take into account how the fact that we are working with individual students constrains participation, as their interactions will be different from those arising in a traditional classroom. For example, when studying autonomously, the student is forced to find solutions and explanations to a far greater extent than if he or she were participating in a classroom, where it is much easier for an answer to appear as a result of a joint effort (Soury, 98).

In the Mathematics Education literature, we find numerous references concerning the communication processes among the participators in the classroom, but they usually attend oral and not written "dialogues" (Sfard et al., 98; Hall, 95; McClain et al., 96; Yerushalm, 97). Contributions related to the interaction via e-mail are not so frequent, althoug important experiences have been made not only in mathematics domain (Trushell et al., 96; Smith et al., 99)

We should exercise caution in applying those considerations about oral conversation to what occurs in a written exchange, where on the one hand the cognitive and metacognitive actions are not the same (León et al., 98), and on the other, the computer imposes itself as an intermediary in the communication process.

The tutorial process

At the beginning of each tutorial session, the student accessed a welcome page that was modified weekly and also in special cases in order to convey additional information. From this page, he was able to link to the web specially designed for him, in which most pages were linked in a linear sequence.

The student was able to work interactively, allowing a continuous exchange with the tutor. For the implementation of the Web pages, we adapted Depover's recommendations for multimedia material design, thus guaranteeing the student an opportunity to regulate his activity autonomously (Depover et al., 98)

During the week before the student left for his athletic training, we began practicing the tutorial process. Our aim was to allow him to familiarize himself with the technological environment. We intended to solve possible problems that could arise due to the lacking familiarity of the student with the multimedia equipment (laptop with modem to conect the internet, and videocamara) as well as to acquaint him with the objectives of the tutorial, and the specific items both parties would commit themselves to. The student knew in this way about the research aspects of the tutorial. In this period, the contact between tutor and student was maintained via e-mail, web board, phone, and video-conference.

During the three weeks that the student was away from high school, we usually spent more than a week on each proposed activity in order to assess the student's difficulties and to be able to offer alternative solutions.

Given the provisional character of the tutorial, we decided to give priority to the student's interest in maintaining the pace of his classmates, and accordingly adapted the contents and manner of presentation employed by the student's mathematics teacher. We designed our activities on the basis of exercises about the geometric visualization of relative positions of lines and planes in two- and three-dimensional space. In this way, from the very beginning we connected to the curriculum proposed by the student´s mathematics teacher, which consists of introducing matrices and determinants as a tool for the solution of systems of linear equations, which subsequently applies to the study of vector geometry.

After concluding the tutorial, we tried to analyze how well the student adapted himself to the school. Also, we looked for signs pointing to reactions of the teacher or student that would not have produced themselves had we not carried out the tutorial.

Methodology

It was the complexity of the situation and the novelty of the experience that drove us to select purely descriptive and qualitative methods of research, with the commitment of coming closer to the reality of both the teacher and the student in a natural way. This required the use of a multitude of techniques to describe the experience rigorously and respect both the teacher and the student, who both faced a totally new situation.

The student we *tutorized* attended a public high school especially suited to the needs of young high-performance athletes. The scheduling of his athletic and academic activities was strongly influenced by his trainers and teachers, whose interests lay in keeping up his academic performance without causing any slack in his training results.

We selected this particular student as an interesting case of study because he often used computer equipment and knew about the Internet and how to navigate. This familiarity with new technologies, his strong commitment during the whole tutorial process and willingness to participate were decisive for the success of this research.

Investigation techniques

We parted from the assumption that directing interviews with teacher and student using previously established questions as a guideline -even open questions- restricts the collection and the selection of the data. The first encounters maintained with the teacher and the representatives of the center confirmed our initial decision to avoid any type of intervention that would cause people participating in the study to use unnatural expressions when speaking or writing. The next paragraph is an extract from the investigation diary written by the researcher:

> *Oriol [the teacher] again makes his reticence toward the questionnaires explicit. He tells me that he doesn't like to fill out questions. Many times he doesn't understand what they are good for, and he believes that not even the students know what to answer. We joke about it [...] I tell him that there are different ways of gathering data and that when speaking with him I also obtain information that I later write down in my diary. He tells me that he prefers this because he feels more relaxed and in this way can talk to me more freely.*

The amount of the investigator's interference regarding each of the people that participated in the tutorial determines the classification of the techniques used, as reflected in Table 1:

Table 1

Relationship between tutor and:	Research techniques	
	Stronger interference	Weaker interference
Student	Inquiring	Classroom observation Participant observation during the tutorial Examination of Web pages
Teacher	Classroom observation Inquiring	Examination of classroom notes
High School		Examination of High School documents

The obtained data were classified as follows:

- A field diary that picked up the investigator's notes of all the interviews with the directive team of the school, the teacher, and the student; contributions and suggestions from other professionals of the university environment; suggestions about the evolution of the study and significative observations about the correspondence via e-mails between the tutor and the student.
- Printouts of the e-mail correspondence between the tutor and the student.
- Web pages designed for the tutorial.
- The teacher's notes from which the individual tutorial sessions were developed.
- Informative documents about the identity of the school that allowed a detailed description of the situation in which we carried out our research.

Data treatment and analysis

For our data analysis, we used the software for qualitative analysis NUD.IST (Richards et al., 94). We started out with only two complementary categories formed by the global intentions underlying the design of the web pages used in the tutorial. The first referred to *content* and consisted of the web pages and e-mail interventions presenting new mathematical contents either explicitly or via an activity. The second consisted of those interventions dedicated to the *planification of the tutorial* in terms of time and technologically oriented aspects. The process of adding new categories emerging during the analysis, search strategies and data cross-referencing were immediately converted to new data in a continuous process of construction and elaboration of ideas.

In the analysis of our data, we followed Tusón's proposal of discourse analysis focusing on a *thematic* dimension, making reference to the content of the participants' intervention and *interlocutive* dimension attending the way in which these interventions are organized (Tusón 99). The analysis of this latter dimension made it possible to describe how the tutor and the student interrelated via e-mail and web pages.

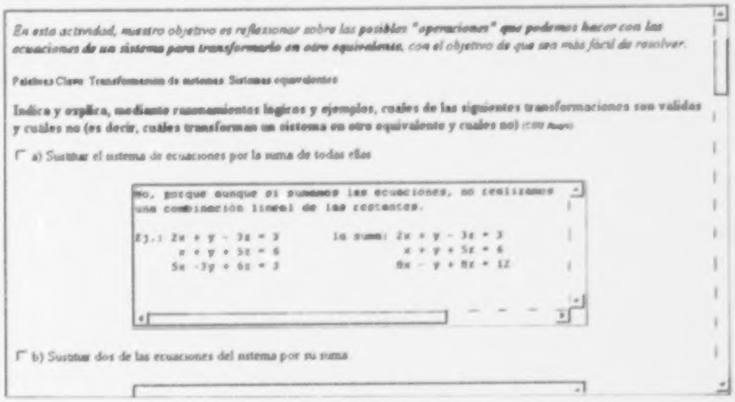
We grouped into a category *Mathematical Content* those e-mail messages and web pages mainly consisting of theoretical contents, exercises, or regulation proposals related to the linear algebra unit we were working on.

Main conclusions and future perspectives

In agreement with the initially proposed objectives and the methodological evolution of our study, we concentrated on the written interactions between the tutor and the student that took place via e-mail. We find that the sequence "Question-Response-Evaluation" considered to be characteristic (and simplest) of student-teacher interactions (Cazden, 88) has to be enlarged by at least two more interactions taking the form of nested loops. These loops are iniciated by the teacher, looking for the participation and creative exploration on the student's side (rows 2 and 3 of Table 2. T means an intervention of the tutor and S one of the student's). These loops repeat up to three times, implying—by the existing lag in communication—that during the whole tutorial process, attention remains centered on one activity. The special properties of written versus oral dialogue call for the formulation of new methodologies allowing an analysis of the complexity of the new sequences. These include widely varying linguistic aspects with respect to the elaboration, the extension, and the wide variety of information contained in one and the same intervention.

We detect interaction sequences of the types showed in table 2 (first and second column). The example printed in the third column of this table shows how the asynchronicity in our *conversation* and the activities involving multiple questions frequently block the sequence after an intervention by the tutor. This block seems to be directly related to the presence or absence of dialog boxes and directly influences how much the student participates offering answers, as well as the quality of his contributions.

Figure 1: Example of one of the Web pages and student's answer into a dialog box



En esta actividad, nuestro objetivo es reflexionar sobre las posibles "operaciones" que podemos hacer con las ecuaciones de un sistema para transformarlo en otro equivalente, con el objetivo de que sea más fácil de resolver.

Palabras Clave: Transformación de sistemas. Sistemas equivalentes

Indica y explica, mediante razonamientos lógicos y ejemplos, cuáles de las siguientes transformaciones son válidas y cuáles no (es decir, cuáles transforman un sistema en otro equivalente y cuáles no) [illegible]

☐ a) Sustituir el sistema de ecuaciones por la suma de todas ellas

```
No, porque aunque si sumamos las ecuaciones, no realizamos
una combinación lineal de las restantes.

Ej.: 2x + y - 3z = 3         la suma: 2x + y - 3z = 3
      x + y + 5z = 6                   x + y + 5z = 6
     5x -3y + 6z = 3                  8x - y + 8z = 12
```

☐ b) Sustituir dos de las ecuaciones del sistema por su suma

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When information and requests for answers or reflections are introduced on a web page using dialog boxes (D.B.), the students stops to elaborate an answer. As soon as the tutor chooses to intervene via e-mail, communication is blocked. One possible interpretation for this is that dialog boxes on the screen enhance the willingness to stop reading and proceed to elaborate an answer. In the case of e-mail, questions are considered as a part of a wider narrative including the whole of the message in question, and the student does not feel as if he were addressed. The participation of the student during the whole tutorial process supports this hypothesis, as his written participation via web pages is far greater than via e-mail.

Table 2

(1) T	Objectives of the activity Multiple Questions Preferences	[Web] *What we are trying to do now is to find the possible intersections between lines and planes in space. Let´s begin with the simplest case, taking two lines in the plane. What possibilities do you think are there for the intersection?* Dialog Box (D.B.) *The next step is to analyze the possible intersections between two planes in space. Could you try to describe them?* D.B. *The third case we'll talk about is that of a line and a plane. What are the possibilities now?* D.B. *The last case could seem to be a little more complicated. How many possibilities are there for the intersection if we take three planes in space?* D.B.
(2) S	Strict response or expression of incapacity to resolve	[Response in every dialog box. We only attend the last question made] *Two parallel planes and one other- intersecting both* *That all three coincide in a point* *That they cross in distinct points* *Three coincide*
(3) T	Elaboration in case of correct response, otherwise help and invitation to answer again Incition to self-supervision	[Web. Presentation of the solution using graphics and illustrative text] *What happened this time, Teo, did you get all possibilities, or did you miss any?* D.B.
Loop T S	Response Emotional reactions Devolution of the answer Back to (2) one or twice	[Web] *The one where two planes coincide and the other crosses, and the one where three planos intersect in a line. ¡Ah! I'm not sure if I put the one with three parallel planes, it was a mistake* [E-mail] *Yes, this exercise was a little more complicated, but you only missed three possibilities. It´s an exercise in geometric visualization that I like very much. The third one you describe (that they cross in distinct points) I don´t understand very well. Could you try to explain again to me which position you meant?*
(4) T	Grading	

One of the concepts frequently used in this unit and especially difficult for the student to understand is that of *linear combinations of equations*. In one of the proposed exercises, the student was asked to compare the solutions of two systems of equations, one consisting of three equations in three variables, and the second one made up of the single equation resulting from summing the three former ones. His stumbling block when reading about a system made up of only *one* equation was the apparent contradiction between its appearance and his "photographic image" of a system of linear equations as a picture containing several equalities, one above the other. The difficulty of helping the student to resolve his doubts is increased because the software he normally uses does not allow him to enter and manipulate mathematical expressions, and he therefore was unfamiliar with mathematical notation when using a computer.

Including materials from different sources and allowing the student to compare and analyze permits us to create a learning situation impossible to achieve under the strict control of a teacher. In consequence, we need to limit the roles of the teacher and tutor in situations of tutorial-at-a-distance: content planification; guiding the student; availability and evaluation. The stimulating and interactive aspects of the computer activity could easily increase the distance between tutor and students shaping a traditional face to face situation. We call for the need of researches focusing in the social structure of the learning activity when using computers and not on the changes operated in individuals.

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ELEMENTARY SCHOOL STUDENTS' STATISTICAL THINKING: AN INTERNATIONAL PERSPECTIVE

Bob Perry, University of Western Sydney Macarthur

Ian J. Putt, James Cook University

Graham A. Jones, Carol A. Thornton, Cynthia W. Langrall, Edward S. Mooney,

Illinois State University

Based on a review of the statistical thinking literature and Biggs and Collis' (1991) neo-Piagetian general development model, the authors have formulated and validated a framework for assessing and fostering elementary students' statistical thinking. The framework comprises four key constructs: describing, organising and reducing, representing, and analysing and interpreting data. The same validation procedures were implemented on two different cohorts: 20 U.S. and 40 Australian students in grades 1 through 5. The data confirmed four levels of statistical thinking for each construct. However, the degree of consistency with respect to the framework was different for the U.S. and Australian samples.

Background

In response to the critical role that data plays in our technological society, international pressure for reform in statistical education at all grade levels has been mounting (Australian Education Council, 1994; Lajoie & Romberg, 1998; National Council of Teachers of Mathematics, 1998; School Curriculum and Assessment Authority and Curriculum and Assessment Authority for Wales, 1996). These calls for reform have stimulated research on statistical thinking, especially in the elementary grades, where, to date, there has been a tendency to focus merely on graphing rather than on broader aspects of data handling and analysis (Lajoie & Romberg, 1998; Shaughnessy, Garfield, & Greer, 1996). Some elements of students' statistical learning have been investigated in areas such as data organisation (Mokros & Russell, 1995), data modeling (Lehrer & Romberg, 1996) and graph comprehension (Curcio, 1987; Friel, Bright, & Curcio, 1997). However, only one study (Jones, et al., 1998) has developed a framework of students' statistical thinking that could be used to inform instruction. To date, the validation of this framework has been limited to U.S. data.

Accordingly, the study reported here adopted a cross-cultural orientation in using data from Australia and the U.S. to:

(a) validate the statistical thinking framework from a more international perspective; and

(b) compare the statistical thinking of Australian and U.S. students in grades 1 - 5.

Theoretical perspectives

The Statistical Thinking Framework (Jones, et al., 1998) is based on previous research (Aberg-Bengtsson, 1996; Curcio, 1987; Friel, Bright, & Curcio, 1997; Mokros & Russell, 1995) and incorporates four key constructs adapted from Shaughnessy et al. (1996): describing data, organising and reducing data, representing data, and analysing and interpreting data. *Describing data* involves finding information explicitly stated in a visual display, recognising graphical conventions, and making direct connections between the original data and the display (Curcio, 1987). *Organising and reducing data* incorporates mental actions on data such as ordering, grouping, and summarising (measures of centre and spread) (Moore, 1997). *Representing data* involves the construction of visual displays including representations that exhibit different organisations of data. *Analysing and interpreting data* involves recognising patterns in the data, and making inferences, interpretations, and predictions from the data. It includes what Curcio (1987) referred to as "reading between the data" and "reading beyond the data" (p. 384).

The Framework is situated in the neo-Piagetian general development model (Biggs & Collis, 1991) that recognises different levels in the complexity of students' thinking. Four levels of statistical thinking across each of the four constructs were hypothesised. Level 1 is associated with idiosyncratic thinking; Level 2 is seen to be transitional between idiosyncratic and quantitative thinking; Level 3 involves the use of informal quantitative thinking; and Level 4 incorporates analytical and numerical reasoning about data.

Method

Three groups of 20 students each form the population for this study. In each case, four students were purposefully selected from each of grades 1 - 5. This selection was based on teacher assessment and student achievement in mathematics, with two students being selected from the middle 50% and one from both the lower and upper quartiles of each grade level. Different starting school arrangements resulted in some variation in the students' ages among the three samples. The U.S. students (mean ages by grade 6.8 years to 10.8) were selected from a midwest school. The first Australian sample (A1) (mean ages by grade 5.5 through 9.7) came from a non-government school in a provincial city in northeastern Australia. The second Australian (A2) sample (mean ages by grade 6.7 through 10.8) emanated from a suburban government school in a large city in mideastern Australia. This school was characterised by its ethnic diversity, with more than 75% of its students coming from non-English speaking backgrounds

The process used to validate the framework for these three samples was similar to that used in the earlier study (Jones, et al., 1998). It involved three components:

(a) interviewing and analysing target students' responses to a Statistical Thinking Protocol[1] which was based on the Framework and comprised tasks from three separate contexts: Sam's friends, Beanie Babies, and a beanbag game between Susie and Pete;

(b) examining the stability of the students' thinking over the four constructs; and

(c) illuminating the distinguishing characteristics of each thinking level.

This cross-cultural validation also compared the statistical thinking of students in the two countries. Qualitative analysis was used to address all three parts of the validation.

The Statistical Thinking Protocol was administered to each student by a member of the research team. Each of these tasks in the Protocol incorporated open-ended questions, and a series of probes. Seven questions were associated with describing the data, seven with organising and reducing data, three with representing data, and six with analysing and interpreting data. Students' responses were audio taped and transcribed, and student artefacts such as drawings and graphs were collected.

To determine students' statistical thinking levels, all researchers used a coding rubric that had been developed in the earlier study (Jones, et al., 1998). They also adopted a double-coding procedure (Miles & Huberman, 1994) in which two researchers in each country independently coded all questions for each student's interview protocol. The coding rubric enabled each question to be coded according to its construct and the level of thinking exhibited by the student. The two researchers then met to compare and negotiate thinking levels on each question. Following this negotiation, a target student's dominant level of thinking for each construct was determined by identifying the student's modal level of thinking for all questions associated with that construct. For the U.S. sample, the two researchers initially agreed on the coding of 66 levels out of 80, giving a reliability of 83%. The equivalent reliability measure for the Australian data was 88%.

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[1] Space precludes the inclusion of this Protocol in the paper. See Jones, et al. (1998).

Results and Discussion.

Tables 1 - 4 show the median levels by constructs within each grade for the three samples of elementary school students. In using the median as the centre for each set of student data, we assumed that the levels data were interval, that is, that the intervals between statistical thinking levels are equal. Given that the levels are consistent with Biggs and Collis' (1991) developmental theory model, this assumption seems reasonable.

Table 1: Describing data displays: Median statistical thinking levels

Grade / Sample	**1**	**2**	**3**	**4**	**5**
U.S.	1.5	1.5	3	3	3
A1	1.5	2	2	3	2.5
A2	1	2.5	1.5	2.5	2.5

Table 2: Organising and reducing data: Median statistical thinking levels

Grade / Sample	**1**	**2**	**3**	**4**	**5**
U.S.	1.5	1.5	2.5	3	2.5
A1	1	1	2.5	2.5	3
A2	1	1.5	2.5	2	2.5

Table 3: Representing data: Median statistical thinking levels

Grade / Sample	**1**	**2**	**3**	**4**	**5**
U.S.	1.5	1.5	3	3	3
A1	1.5	1.5	3	3	3
A2	1	1	3	3	2

Table 4: Analysing and interpreting data: Median statistical thinking levels

Grade / Sample	1	2	3	4	5
U.S.	1	1.5	2	2.5	3
A1	1.5	2	2	2	2
A2	1	1.5	1.5	2	2

Inspection of Tables 1 - 4 reveals that:

(a) the median thinking levels on all four constructs for the U.S. and Australian samples generally increased or remained constant with increasing grade levels, although this was not always a smooth transition, especially in the A2 sample;

(b) the median levels for the U.S. and Australian samples in grades 1 and 2 were similar on most constructs with the Australian samples being slightly higher by grade 2 on describing data displays;

(c) the median levels for the U.S. sample in grades 3, 4 and 5 were at least as high or higher than their Australian counterparts in the same grades; and

(d) the A2 sample generally performed at a slightly lower level than the other samples.

The overall profiles for the U.S. and Australian students show similar growth patterns and trends. Most of the differences between the data of the two countries may be attributed to age differences between corresponding grades, different curriculum emphases at each grade (for example, sample A2 does not meet probability in its curriculum, perhaps significantly decreasing the students' potential exposure to data handling while sample A1 has had little formal instruction in measures of centre, which are not studied in depth until grades 6 and 7), and sampling differences, such as the high level of non-English speaking background students in the A2 sample.

With respect to stability across the four constructs, inspection of the median levels at each grade level reveals a high degree of correspondence for the U.S. and A1 data with greater inconsistencies in the A2 data. When the raw data (not included) for all 60 target students were examined, it was found that 38 (63%) of the students exhibited the same level of thinking on at least three of the four constructs. The consistency figures (at least three levels equal) for the samples were: U.S.: 80%, A1: 65% and A2: 45%. Apart from A2, the consistency data were similar to earlier studies (Jones, et al., 1998) and indicated that the levels are relatively stable and coherent across all four constructs. In the case of the A2 sample, there are a number of language and cultural issues which need to be considered when interpreting the data.

Analysis of the students' statistical thinking across all three samples illustrated the distinctive features of each level of the Statistical Thinking Framework. Students exhibiting Level 1 thinking were narrowly bound to idiosyncratic reasoning that was often unrelated to the given data and focused on their own personal data banks. Level 2 thinkers were beginning to recognise the importance of quantitative reasoning and tried to make sense of the data. Students exhibiting Level 3 generally used quantitative reasoning as a basis for statistical judgments and had begun to analyse data from multiple perspectives. Level 4 thinkers used both analytical and numerical reasoning in data exploration. Moreover, they showed evidence of being able to make connections between different aspects of the data. Some student examples concerning thinking about measures of centre further illustrate these levels.

Level 1 - Idiosyncratic. In response to the question "About how much did Pete score each day?", a Year 1 student from the A1 sample said "Ten ... maybe he likes ten", while another from A2 suggested "Some days - on Monday, Tuesday, Wednesday, Friday and Thursday, they go home and play after school". When asked "About how many friends would you expect to come to Sam's place each week during the summer holidays?, two Year 1 students from the A1 sample suggested "Two ... because his Mum wouldn't be so busy" and "One ... because ... like you're having too much friends over".

Level 2- Transitional. In response to the question "About how much did Susie score each day?" a Year 5 student from the U.S. sample moved towards using the mode and said "It is mostly 3 and 4". Similarly, In response to "How many Beanie Babies does each child have?", a Year 5 student from the A1 sample explained that "they should have around 4 or 5 or 3 ... cause if the kids are little, instead of having really hard toys and hurting themselves, they all have soft toys".

Level 3 - Quantitative. In response to the question about Susie's daily score, a Year 2 student from the A2 sample concentrated on the mode and said "3, 3 is the most number in there". In contrast, when asked, in another part of the protocol, the average number of children who came to visit Sam each day, a Year 4 student from the U.S. sample appeared to balance the numbers in an effort to approximate a mean: "About 3 or 4. This one has 3, this has 4, this has 7. So if you take 3 away from that [the 7] and give it to the day with 0, you have about 4".

Level 4 - Analytical. In response to "What was Pete's average score?", a Year 2 student from A2 said "Made the average, plusing all together and divide by 5 - I learned that at Vietnamese school". In contrast, a Year 5 student from the U.S. sample, responding to a question about the average number of Beanie Babies for each child, said "Well, if you take this and give it to Amy, and this one from here and give it to Amy, and this one and give it to Amy, and then one to Ben from here, then the average would probably be 3 because, well, they would all have 3. That shares them out".

Educational Importance of the Study

There have been numerous calls to extend the research base on students' thinking as a means of informing instruction (e.g. Fennema, et al., 1996). This study addresses the void in research-based knowledge of students' statistical thinking. Moreover, in accord with Shaughnessy's (1992) recommendation, we investigated students' statistical thinking across different cultures. Sufficient similarities in the students' levels of statistical thinking and consistency across levels within this thinking were demonstrated to suggest that the Framework for Statistical Thinking is applicable not only in the U.S. but also in varied contexts in Australia. These results imply that curriculum developers and teachers from different cultures may find the framework useful for informing instruction and assessment in data exploration and analysis.

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THE PROPERTIES OF NECESSITY AND SUFFICIENCY IN THE CONSTRUCTION OF GEOMETRIC FIGURES WITH CABRI

Angela Pesci
University of Pavia, Department of Mathematics

Abstract
This work studies a specific aspect of the use of Cabri – Geometry as a mediator between the space-graphic level and the theoretical level in the learning of geometry. Follow the experiences carried out with students of primary and secondary school, the need to deepen the understanding of some problems linked to the process of constructing figures with Cabri, was evident. This paper is a contribution in that direction, dealing with some specific questions connected with the meaning, in Cabri environment, of the feature of 'sufficiency' or 'necessity' for the properties of a figure. What is here described is important if one wants to be able to observe, analyze and evaluate the process set in motion through Cabri in relation to these specific geometrical features and, in addition, if one wants to take into consideration a possible use of Cabri in class to develop students' geometrical thinking.

Introduction

The literature on the use of Cabri – Geometry software as a possible mediator between the space-graphic and theoretical levels, more precisely between that which one sees perceptually on the screen (for example a constructed figure) and that which one knows of the same figure in terms of theoretical geometric relations, is by now very ample. It has already highlighted some fundamental ideas, as for example, the centrality, in this environment, of the dialectic between design and figure and between figure and concept (Laborde C. and Strasser R., 1990, Laborde C., 1993, Laborde C. and Capponi B., 1994, Holzl R., 1995, Mariotti M. A., 1995, Lanzi E. and Pesci A., 1997, Pesci A., in press), between construction of a procedure and its theoretical justification and between argumentation and demonstration (Arzarello, F., Gallino G. et al., 1998, Arzarello F., Micheletti C. et al., 1998, Arzarello F. and Mariotti M. A., 1999, Laborde C., 1997, Mariotti M. A. et al., 1997, Mariotti M. A., 1998, Noss R. et al., 1994).

The quoted research has also put in evidence the important fact that the passage to the theoretical level through the use of Cabri is not spontaneous, but is only the fruit of precise didactic engineering.

In this context, following experiences carried out with 10-14 year old pupils and the reflections brought out by our teachers group (Lanzi E. and Pesci A., 1997, Bardone L., Lanzi E. and Pesci A., 1998, Joo C., in press), the need has arisen to confront, in depth, some questions connected to the process of construction of a geometric figure with Cabri. (All our experiences refer to Cabri I, MS-DOS, version 1.7, but what follows is also valid for other versions.)

Here, in particular, I will try to respond to the following specific questions: during the construction of a figure with Cabri, when is a property utilized with the connotation of "a necessary property"?, when with the connotation of "a sufficient property"? And, when do some properties manifest themselves as necessary and sufficient for the making of the figure? In what sense do 'necessary and sufficient' have to be interpreted?

Responding to these questions is very important if one wants to be able to observe, analyze and evaluate the process that Cabri implements in relation to the identification of several specific aspects of geometry. It is also important if one wants to take into consideration a classroom use of

Cabri as an opportunity to develop the geometric thinking of the children in the direction of a theoretical reflection on the geometrical properties that characterize a figure they have already seen.

The reflections shown here are a contribution towards an interpretation of the process which the students implement and of the difficulties which they might encounter when they have to construct a figure using Cabri; a process which must inevitably take into account the meaning which the various geometric properties assume in relation to the perceptive image of the same figure.

In reference to the functioning of Cabri, here one must simply remember that, to construct a figure in this environment means carrying out a construction which passes the "dragging" test. In other words, it is necessary to produce a construction after having rethought the characteristic geometric properties of the figure and then subjecting the basic elements of the construction to dragging, to check that, also during movement, the constructed figure maintains the desired characteristics.

All of this, of course, redirects the attention both of the student and of the teacher, to the process that was followed in obtaining the figure and in particular to the modality with which the geometric properties were taken into account. From the didactic point of view, this allows planning rather interesting activities, but also requires, on the part of the teacher, the awareness of what the interconnection between the geometric competence of the student and his interaction with Cabri is and how it develops.

The objective of this presentation is therefore twofold: on one hand I intend to analyze the process of construction of a figure in a Cabri environment, identifying, in the various phases, the recourse to some specific geometric aspects (the properties of a figure with the connotations 'necessary' and 'sufficient'). On the other hand, I would like to offer a contribution of a theoretical nature with some "spin-off" possibilities on the level of didactic engineering.

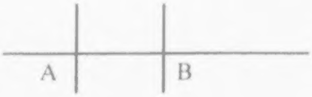
2. Necessary properties, sufficient properties, necessary and sufficient properties

In this, and the two following paragraphs, I will refer to a hypothetical user of Cabri: it may be either the teacher or the student. Here it is not important to distinguish one from the other since I would like to characterize, in the direction already mentioned, interaction with Cabri.

Let's suppose that we have to construct a square with Cabri. We will examine in detail only one of the possible procedures, chosen from among the simplest ones, with the goal of best highlighting the attribution, in some specific constructive phases, of the connotations of necessity and sufficiency to the geometric properties involved.

We might begin by creating a straight line through two points and then constructing through each one of them the perpendiculars to the line:

Fig. 1

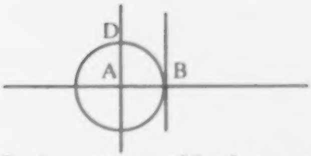


In this phase, points A and B are already thought of as vertices of the square and the two perpendiculars at A and B have been constructed because they are considered to be necessary.

If we call P_1 the property of having two right angles, since we maintain that P_1 is necessary for having a square, then it is imposed in the constructive procedure by means of the two perpendiculars. P_1 is necessary in the sense that it is shared by all squares: if ABCD is a square, then, for ABCD, P_1 is valid.

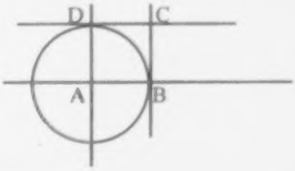
At this point, since the two consecutive sides of the square are equal, it is possible to construct the circumference with center A and passing through B and to choose as point D one of the two intersections of the circumference with the perpendicular through A. Thus for example:

Fig. 2



In this phase, if we call P_2 the property of having two consecutive equal sides, property P_2 is therefore considered as another property necessary for having a square. To conclude the construction, one more step is needed; for example, the construction of a final perpendicular, the one through D to AD.

Fig. 3



Therefore, property P_3, which is that of having a third right angle, is also considered to be necessary to obtain a square.

At this point, the square ABCD appears on the screen and it passes the validation test: when the 'base elements' are moved, in our case A and B, ABCD always remains a square, big or little "as much as you want" (limited, obviously, by the screen).

We observe that the properties P_1, P_2, and P_3, can be summarized in the following two properties: having three right angles and two consecutive equal sides. Since a stable square has been obtained, it can be said that they are sufficient to have a square. Because they were chosen, their necessity being recognized, they assume therefore the connotation of necessary and sufficient properties for the square.

To synthesize the constructive procedure described, when constraints are imposed, we think of some properties which are necessary for the square, when a square appears on the screen and it proves to be stable, we have the proof of the sufficiency of the conditions imposed. On the basis of the construction examined, a square can be defined as a quadrilateral with three right angles and two equal consecutive sides. We note that they are two geometric properties which are independent one from the other as it is easy to prove with or without Cabri (any rectangle has three right angles but not two consecutive equal sides and any rhombus has two consecutive equal sides but it doesn't have the right angles). We can say therefore that the definition refers, in the strict sense, to necessary and sufficient properties.

Let's remember that, in general, the propositions $P_1, P_2, \ldots, P_n$ are independent if none of them can be demonstrated by means of the others (each is independent of the set of the others). We observe that having necessary and sufficient properties available for a figure, in the strict sense, does not only constitute an 'elegant' requirement from a theoretical point of view but, as well known, it is fundamental in normal geometric activity because it gives indications of the requirements that must be checked to be certain that it deals with a given figure (and it is clear that it is important to minimize the number of controls).

Therefore, more generally, we observe that if we begin to impose necessary conditions for a figure, their sufficiency is proved as soon as they pass the dragging test. In such case, the conditions imposed constitute a set of necessary and sufficient properties for that figure and what is interesting is that they are usually necessary and sufficient properties in the strict sense. In effect, with Cabri, imposing necessary but redundant properties is not spontaneous, (for example, four right angles to make a rectangle) because as soon as the properties are sufficient, the desired figure appears on the monitor and then one can go on to validate it.

3. The case of necessary but not sufficient conditions

In some situations it is very simple to verify the insufficiency of the conditions imposed during the constructive procedure of a figure. For example, in the preceding paragraph, in the first two phases of the construction (Fig.1 and Fig. 2), since the square has not yet appeared on the video, it is clear that something is missing from the constructive procedure of a square. The imposed conditions (two right angles in the first phase and the addition of two consecutive equal sides in the second phase), even if necessary, are not at all sufficient to have a square. In effect, only in the third phase (placing the last right angle) is the square finalized. In this case therefore, the insufficiency of the conditions imposed is visually evident.

In other situations instead, it is only during the figure's validation phase that one is aware of the insufficiency of the constraints imposed. For this reason, let's examine the construction of a rectangular trapezium. One can begin with a segment and a straight line perpendicular to it at one end.

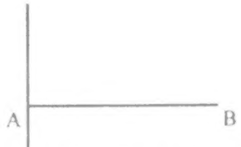


Fig. 4

Having then chosen a point D on this perpendicular, the parallel to the segment can be traced and choosing a second point C on this parallel, the fourth side of the trapezium can be drawn.

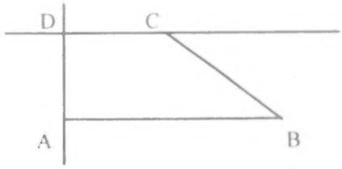


Fig. 5

To obtain this construction, a condition of perpendicularity and one of parallelism have been established (the second right angle, as the figure shows us and the geometry confirms, is a consequence of the parallelism imposed: if two lines are parallel, a line perpendicular to one of them is perpendicular also to the other).

At this point, since a rectangular trapezium appears on the video, the construction seems to be finished. Nevertheless, if one goes on to validating it (the base elements are A and B which are free, and C and D each one tied to a straight line), it can be easily seen that it is possible to obtain the following figure

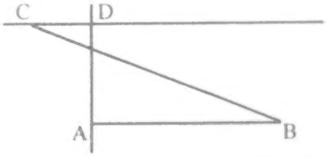


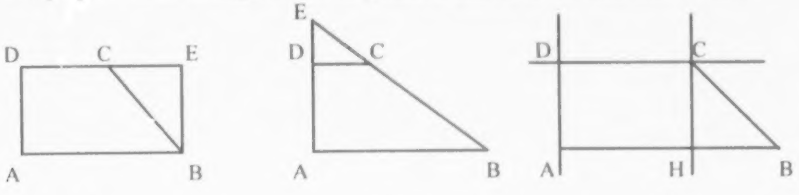
Fig. 6

which we are certainly not ready to consider a rectangular trapezium.

We observe that the quadrilateral ABCD maintains, during the validation phase, both of the constraints imposed (a right angle and a pair of opposing parallel sides) and the second right angle (which we have deduced as 'theorem'), nevertheless the property of convexity, which is by all means necessary for a rectangular trapezium, is not deducible from the construction created, as the counter-example of Fig. 6 well shows.

Therefore, it is necessary to carry out a procedure which assures, for our figure, the property of convexity and this can easily be done, for example, if first a rectangle is constructed or a right triangle and then, inside, the desired rectangular trapezium. Or, it can be done if one starts with a segment and a perpendicular through one of its points as shown in the following figure.

Fig. 7



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With reference to the first figure, C is tied to the segment DE. With reference to the second, D is tied to the segment AE and in the third, point H is tied to the segment AB.

The construction of the rectangular trapezium therefore shows that establishing the non sufficiency of the properties imposed to obtain a figure is not always immediate. Moreover, it gives us a good opportunity to underline the necessity of a forgotten property, convexity. The simplicity with which Cabri supplies the counterexamples cited helps create geometric reflection in that direction.

The following is a final observation in this respect. In the preceding construction of the square, or in other constructions (for example if a rectangle is constructed with three perpendicular relationships or with two parallel relationships and one perpendicular or a parallelogram is obtained with two parallel relationships) the property of convexity, albeit necessary for these figures, is not explicitly required. Nevertheless, all of the constructions cited, give convex figures. This means that, in each of these cases, the property of convexity is a consequence of the properties imposed; that is, it derives from these as a theorem. (Note: also in the construction of the rectangular trapezium it was not explicitly imposed, because there is not such a condition in Cabri, but it was necessary to take care of it explicitly to avoid 'interwoven' figures.)

It is evident, from all this, how effective a construction activity with Cabri can be in initiating a meaningful geometric discussion even at the middle school level.

4. The case of sufficient but unnecessary conditions

Sometimes, it can happen that a figure is constructed imposing sufficient but not necessary conditions and then obtaining special cases for that figure. Awareness of those 'particularities' can come more or less immediately as can be seen in the following examples.

If one wants to construct a rectangle and begins with a circumference and makes two perpendicular diameters, it is clear that the result is a square. that is, a very special rectangle.

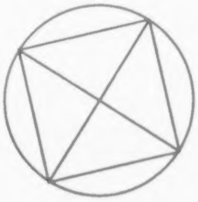

Fig. 8

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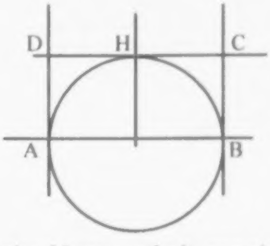
In effect, the condition of being perpendicular is not necessary for a rectangle (the equality of the diagonals is, however).

Analogously, if one constructs a rectangle, again starting with a circumference, one of its diameters, the perpendiculars at its ends, and a parallel through point H (see Fig. 9),

Fig. 9



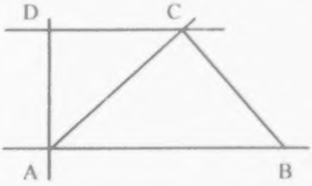
another special rectangle is obtained because, in it, one side is the double of the other; a condition which is certainly not necessary for a generic rectangle.

In these examples, it is evident, also visually, that special cases of the desired figure have been obtained. The geometric exploration that could follow, also proves to be rather simple.

Let's instead consider the following rectangular trapezium constructions.

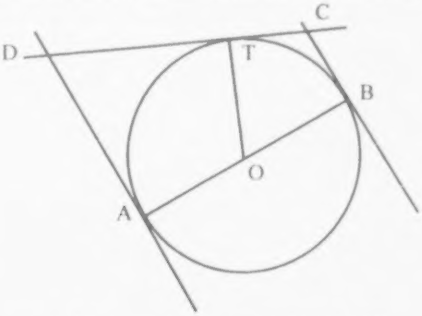
a) Starting from a straight line through A and B, the perpendicular line to it is created through A; choosing a point D on this perpendicular, the bisector of the angle $B\hat{A}D$ is traced and it intersects the parallel to AB through D.

Fig. 10



b) Starting with a circumference with center O and passing through B, traced the perpendiculars to AB at A and B, a point T is chosen on the circumference, the perpendicular to OT is traced at T, and points C and D are obtained as intersections with the two original perpendiculars.

Fig. 11



In both cases stable rectangular trapeziums are obtained. Nevertheless, the proposed constructions give rise to special figures.

In case a) it is evident that AD=DC and therefore, rectangular trapeziums with the height equal to one of the two bases are made. This is a condition which certainly is not necessary in any rectangular trapezium.

In case b) rectangular trapeziums which are circumscribed to a semi-circumference are obtained and this condition also is not necessary for a generic rectangular trapezium. In another way, we observe that, in effect, in the construction carried out, OT=OB=OA and therefore there exists a point, on AB, which is equidistant to the other three sides of the trapezium. This is a property which generally does not occur in rectangular trapeziums and therefore is not necessary. A third way to establish (equivalently) the uniqueness of the construction carried out is to observe that CT=CB and DT=DA (properties of the tangents) and therefore, CD=CT+TD=BC+AD. That is, in our trapezium, the sum of the bases is equal to the oblique side; an unnecessary condition for any rectangular trapezium.

In these two cases, realizing that the figures constructed are 'special' is a little more complex and the geometric arguments are relatively demanding. Here as well, the contribution of Cabri could be notable because, in each of the two cases, the production of some opportune counter-examples using Cabri could prove the 'non-necessity' of the geometric relationships mentioned. That is, rectangular trapeziums that do not verify the relationships of equality between the segments cited. Obviously, this investigation should be carried out using different constructions, for example chosen from among those proposed by classmates; the recourse to the function 'measure' available in the menu could also be of some aid (also if preferred use could be made of 'impressive' counter-examples in which the 'measure' function would be completely useless).

Although this constitutes a rather demanding activity, it could be considered interesting as an opportunity to develop geometric thinking and, in any case, should be born in mind if a class activity is planned designing a figure with Cabri.

5. Final Observations

On the basis of what has been shown, it can be said that in the initial choice of the geometric properties from which the designing of a given figure begins, that is, in the selecting of the necessary properties for that figure, the geometric knowledge of the designer is fundamental. Cabri then highlights when such properties are also sufficient for the same figure and this could be the source of (geometric) reflection for the user. It can verify that a property, which one intends to impose, is a consequence of those already imposed (and in this case the deduction of such a property is a theorem which has as its premise the properties already imposed). Or, it can happen that it is necessary to add constraints to a construction, which one wanted to complete, because the counter-examples which have appeared on the screen show the insufficiency of the conditions imposed (the case of convexity for the rectangular trapezium).

With Cabri, therefore, it is easy to establish (even by means of counter-examples) the insufficiency of necessary properties. It could be more difficult, on the other hand, to establish the non-necessity of sufficient properties because in this case more complex geometric considerations can intervene (as seen in the case of the trapeziums of Fig. 10 and Fig. 11).

The theoretical analysis conducted could also have consequences for the didactic plan in relation to the type of situation described. It is shown that possible reflection on the properties of a figure, as 'necessary properties' should be referred to the initial constructive phase of the figure, while the final constructive phase (dragging included) is connected to the reflection on the 'sufficient' properties of the same figure.

The choice, by the teacher, of the geometric figures to be built using Cabri, for a contingent didactic plan, should be made keeping in mind both the problems which such a construction brings with it and the didactic opportunities which such situations can offer. Here the understanding is that the teacher intends to exploit such a constructive activity to aid the effective development of the geometric thinking of the children.

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STUDENTS' PROCESSES OF SYMBOLIZING IN ALGEBRA:

A SEMIOTIC ANALYSIS OF THE PRODUCTION OF SIGNS IN GENERALIZING TASKS

Luis Radford
Université Laurentienne
Ontario, Canada

> OK. You can say …you make ... OK you add the figure...oh my God, how do you say it [in algebraic symbols]… The figure plus the next figure? (Annik, Grade 8 Student)

Abstract: *In this paper we present some results concerning the students' production and use of signs in the elaboration of the general term of a pattern. Considering the students' production of signs as a process embedded in the activities that the signs mediate, the investigation reported here focuses on a semiotic analysis of the students' strategies seen as a set of goal-oriented heuristic actions displayed by the students in the attainment of the objective of the generalizing activities. The analysis was carried out in terms of a two dimensional grid whose purpose was to shed some light on two key elements in the mediating role of signs. The first one concerned the meanings with which signs were provided by the students. The second centred on the manner in which the students semiotically articulated the relation between the general and the particular. The results (conducted through an interpretative protocol analysis managed with the NUD·IST program for qualitative research) suggest that novice students tend to conceptualize signs as indexes (in Peirce's sense) having a range of specified indexical meanings supported by different views of the relation general-particular.*

1. Framework and Preliminary Remarks

In our ongoing longitudinal research program about students' processes of symbolizing in algebra[1], we are tracking a cohort of students for three years in order to understand their acquisition of the algebraic language. By students' processes of symbolizing we mean the ways students understand, produce and use signs. Our interest in investigating the students' processes of symbolizing in algebra is related to the need to better understand the difficulties that novice students usually encounter in mastering the algebraic language –difficulties systematically reported in literature since the pioneer studies of Davis (1975) and many other studies conducted in the 80's (e.g. Matz (1985) and Kieran (1989)) up to the more recent works (e.g. MacGregor & Stacey (1997) and Kirshner (in press)).

Our work is embedded in a theoretical perspective which puts forward the intimate epistemological link between signs and thinking as stressed in Vygotskian approaches to the mind. While, in most of the analytical and structural traditions in the philosophy of language, signs appear as *aiding things* to think, Vygotskian and some recent socio-cultural and anthropological approaches, in contrast, attribute to signs a constitutive epistemological role in that signs are seen as external cultural 'tools' *imbricated* in and *integrated* into the individual's conceptual functioning. Furthermore, the production of signs, according to our framework, is dialectically related to the activity (in Leontiev's sense) that the signs mediate. In this line of thought, we dealt, in a previous work (Radford 1999), with the different meanings with which

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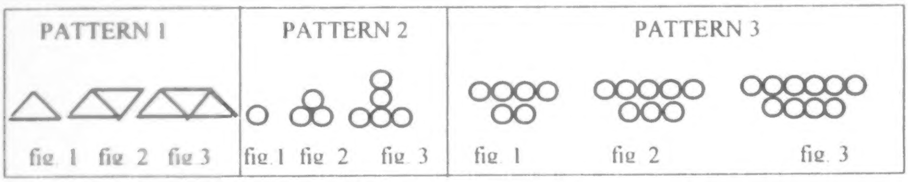
students provided signs in order to *understand* them in the algebraic context of generalization of patterns. In this paper we want to go further and to investigate in a more precise manner the students' *production* and *use* of signs in activities whose objective is the elaboration of a symbolic algebraic expression for the general term of a geometric-numeric pattern. Our approach takes into consideration a specific semiotic problem related to the construction of the general term of a pattern: a problem of *denotation* which can be stated as follows. Since the different elements of a pattern are characterized by the ordinal position they occupy in their well ordered sequence, the elaboration of the expression for the general term of a pattern requires that such a term be referred to a position which cannot be arithmetically expressed (at the modern algebraic level, this position, of course, is commonly denoted by 'n' or another single sign-letter of the alphabet or even a more complex assemblage like u_n). We shall call this particular denoting act 'indexing'. From a cognitive point of view, the indexing act brings forward several problems related to (1) the choice of the indexing signs, (2) the meaning of the indexing signs and (3) the way in which the conceptual-semiotic relation between the general and the particular is ideated. The problems posed by the indexing act in expressing generality are attested to by the history of mathematics, where we find different conceptions about the way the particular and the general are related. These conceptions, of course, are culturally framed as is the choice of the sign systems to express generality –sign systems of which the history shows us a rich variety, such as segments and non positional letters like η, λ, μ, χ in Antiquity (see Radford 1995, p. 47 ff.) or some 17th and 18th century AD additively based sign systems such as x', x'', x''', x'''', x''''' (see Radford 2000). The problems arising from the indexing act are also attested to by the difficulties that contemporary students encounter when trying to elaborate general expressions in patterns. The investigation of the semiotic nature of these students' difficulties is the purpose of this paper.

2. Methodology

The general methodology of our longitudinal research program was sketched in Radford (1999, in press). For the purposes of this paper let us mention that the students of the 4 classes that we are following up for three years worked on activities which included (among others) the three patterns given below. They worked in small groups (usually comprised of 2 or 3 students) and, at the end of the activities, the teacher conducted a collective discussion. Before asking the students to find an expression for figure n, they were asked to perform an arithmetical investigation (e.g. to find how many circles are in figure 10, figure 100).

The data mentioned in this article come from the first year of the field research (1998-99, when the students were in Grade 8, i.e. in their very first year of learning symbolic algebra). The data was processed following an interpretative, descriptive protocol analysis (details in Radford 2000) and was managed using the *Non-numerical Unstructured Data Indexing Searching and Theorizing* (Nud·Ist) program for qualitative research. The analysis of the production and use of signs in generalizing tasks was conducted in accordance to the *goal* of the activity as it was devised by the students. The *goals* (in Leontiev's sense 1984, p. 113 ff.) gave rise to three main categories of actions which oriented the students' heuristic processes in the attainment of the objective of the activity, namely, the construction of the general term of the pattern. We called the heuristic oriented actions 'strategies' and examined them in light of a two dimensional grid whose axes are related to the understanding of the mediating role of signs in the accomplishment of the activity, as pointed out in the framework discussed in Section 1. The first dimension concerned the meanings with which the students provided signs in the indexing act of denotation. The second dimension focused on the manner in which the students perceived the semiotic relation between the general and the particular. Section 3 deals with the description of some of the students' strategies. These strategies have already been mentioned in one way or another by other researchers in previous works on the learning of algebra. Our contribution is to be found at the level of the semiotic interpretation that we offer for the strategies –something which we do in Section 4– as well as at the level of the conclusions that we reach in Section 5.



3. Description of students' strategies

There are three important strategies followed by the students when they try to elaborate a symbolic algebraic expression for the general term of a geometric-numeric pattern. It is important to notice that in solving the problem, the students did not necessarily keep the same strategy. In the course of the activity the goal to reach the foreseen objective could change and so the actions and the whole heuristic process. The reasons leading to a change of strategy are usually related to the interaction between students and between students and teacher. Even though they are very important, they will not be considered here for the limitations of space. However, the reader may consult Radford (1999, 2000, in press).

Strategy 1: The first one is based on the idea of formula as a procedural mechanism in which letters (say 'n') are seen as the designation of a place to be taken by numbers. The usual general heuristic procedure is based on a kind of quasi-trial-and-error method which can sometimes become sophisticatedly controlled but whose success will depend on the complexity of the pattern.

Example 1 (P38A1Ca): (The code of the examples refers to their Nud·ist identification only)

This point can be illustrated with reference to the classic toothpick Pattern 1. In one of our Grade 8 classroom groups, the students found the formula '$f \times 2 + 1$' (where 'f' stands for 'the number of the figure'). When asked by the teacher to explain why they added '1' to '$f \times 2$', the student who proposed the formula said: "Uh...because it works!" and proceeded to show it through many numerical examples.

Example 2 (P18A2Co): Another example, concerning pattern 2, is the following.

1. Madeleine: But, no, that would really be the number of the figure times 2 minus 1. Because, look! 2 times 2, 4, minus 1, 3. One times ... (...) Yes. That would work! 1 times 2, 2, minus 1, 1!
2. Carole: 1 times ... All the time, times 2?
3. Paul: Yes.

Strategy 2: The second general strategy consists in finding a general expression on the basis of certain numerical facts occurring between some terms of the pattern. In Pattern 1 and 2, it was often noticed by our students that the number of ' basic elements' (i.e. circles or toothpicks) in the next figure was two more than in the previous figure. In modern notations, this corresponds to the recursive formula $u_{n+1} = 2 + u_n$. Here is an example:

Example 3 (P18B1Co):

1. Jessy: Look ... n plus two ... (points to a place on the page) This is n. This plus two equals this. This plus two equals this. This plus two equals this. It's n plus two.

The problem with Jessy's recursive formula, applied here to Pattern 2, is, as Michelle noticed, that the formula does not provide one with the total of elements in the figure:

2. Michelle: But, if you want ... if you want like the figure 200 ? (...) But, you want the figure 200, then they tell you n plus two equals the figure 200.
3. Jessy: Yes, it's 198 + 2 ... You would say that the figure before it is 198.
4. Michelle: How do you know that?

The arithmetic experience led sometimes the students to observe other numerical facts. For instance, concerning Pattern 1, in some groups it was noticed that the total of toothpicks in a figure equals the number of the figure plus the number of the next figure. Here are two examples:

Example 4 (P38A1Ca):

1. Guy: (*interrupting*) one plus two, two plus three, three plus four, four plus five, ...
2. Joe: (*interrupting*) five plus six. Oh! (realizing that Guy's idea works) O.k. (inaudible)

Example 5 (P18B2Co):

1. Josh: It's always the next one. One plus two, two plus three (...), three plus four ....

Of course, in these two examples, the students do not state the noticed numerical regularity in a general verbal form. Their understanding and shaping of the general occur at a numerical level. As a matter of fact, the analysis of the protocols shows clearly that students tend to talk about the general through the particular. As we shall see in the next section, this is crucial to the students' ways of symbolizing and expressing generality in symbolic language. Let us now turn to the third strategy.

Strategy 3: This strategy is based on the shape of the figures in the pattern. The main idea is to count the basic elements in each of the structural parts of the pattern and to combine the partial totals into a kind of grand total. In Patterns 2 and 3, the procedure consists in finding the total of circles in each of the branches of a figure and then to add those totals. The next example is related to Pattern 2.

Example 6 (P18B2Co):

1. Judith: If it's the figure it'll always have the number ... like if we say it's figure 12, you'll have 12 on the bottom and then you'll have one less on top vertically.

When applied correctly, this strategy usually leads to the expression n +(n-1) for figure n in Pattern 2 and (n+1) + (n+1+2) or (n+1) + (n+3) for figure n in Pattern 3. The reasons why the students do not go further and group similar terms is related to the meaning with which they provide signs, as we will see in the next section.

4. Semiotic analysis of the strategies

Strategy 1.The production and use of signs in the three strategies presented in the previous section are underlain by different meanings that students ascribe to signs. In Strategy 1 signs are understood as a place to be taken by numbers; that is, they appear as parts of chains of operations functioning as mere emplacements where concrete numbers come to be logged in order to produce a numerical result. From the point of view of denotation, the sign is understood as denoting the number of the figure (see Example 2, line 1). But the arithmetic operational context framing the formula in which signs appear makes signs play another role: that of marks indicating the result of operations. Thus, in the discussion of Example 2, when it came time to put the formula into symbols, the discussion revolves around whether 'n' also has to be included in the total. The students say:

Example 2 (continuation):

1'. Carole: "n", after this, bracket, n times 2, minus 1.
2'. Madeleine: Equals "n".
3'. Carole: You don't have to write equals "n". Do we?
4'. Madeleine: Yes. You have to write it.
5'. Carole: Just ... we don't need "n".
6'. Paul: You need a formula.
7'. Carole: OK. "n" bracket "n" times 2, minus 1. (And she writes $n = (n \times 2 - 1)$)

The syntax of the final formula sheds further light on the actual meaning of signs. Instead of writing "$2 \times n - 1$" as it would be more in tune with the canonical syntax of the algebraic language, the students write "$n \times 2 - 1$". Why? The reason is that the sign "n" in the expression "$n \times 2 - 1$" is an *index* (in Peirce's sense). That is, "n" is pointing to the verbal utterance "the number of the figure times 2 minus 1" in line 1 of Example 2 shown in the previous section. (The dialogue analyzed in Radford 1999 exhibits also this same phenomenon). Of course, the sign "n" indicating the *total* in the students' formula is also an index –although with a different indexical meaning. At any rate the common indexical nature of the two signs "n" guarantees their common appearance in the same symbolic expression.

As to the relation between the particular and the general, it is framed by the operational conception of the symbolic expression. As a result, the particular (v.gr. the prior arithmetical investigation of some concrete figures such as figure 10, figure 100) plays a little role (if any) in informing the form and structure of the algebraic symbolic expression for the general term of the pattern. Hence, when confronted with the question about how many circles in total figure 10 in Pattern 2 has, Carole suggested to find out the general formula first and then to apply it to figure 10 when she said: "If we figure out the formula first then we calculate it, that would be easier than just thinking." The relation particular/general is very restricted in that, on the one hand, the particular serves only to check the validity of the symbolic expression; the general, on the other hand, appears as making the economy of the analysis of particulars.

Strategy 2. In this strategy signs have a different semiotic function. They have to express numerical regularities involving two or more terms which need a semiotic articulation. The difficulties arising here may become very complex in terms of denotational requirements, as clearly illustrated in the next example:

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Example 5 (continuation):

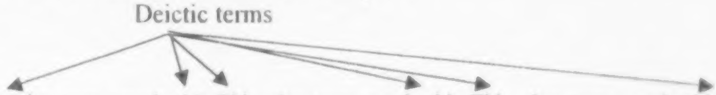
1'. Annik: OK. You can say ... you make...OK you add the figure ...oh my God, how do you say it [in algebraic symbols]... the figure plus the next figure?

In Example 3, as we saw, Jessy's 'recursive formula' was stated verbally as: "... This plus two equals this ...") and a symbolization was also provided: "It's n plus two". When the students came back to this recursive formula after failing to find a non-recursive, direct one, they continued:

Example 3 (continuation):

1'. Jessy: It's always n + 2.
2'. Michelle: Yes, Yes. Sure.
3'. Jessy: n + 2 equals figure 5.
4'. Michelle: n + 2 ...
5'. Jessy: Figure 4. It's like figure 4 + 2 equals figure 5.

Here Jessy proposes to symbolize such a numerical fact as 'n+2'. Thus, at an implicit level, the sign 'n' is seen as denoting the number of circles in a figure which remains unspecified. We see how the 'indexical problem' and the denoting process in which such a problem is embedded would require the differentiation of referents. It is important, in fact, to distinguish: (i) the figures, (ii) their position in the pattern and (iii) the number of circles that they have. Natural language equips the students with a whole arsenal of deictic terms that Jessy indeed exploits to his advantage in line 1 of Example 3:



1. Jessy: (...) This plus two equals this This plus two equals this.This plus two equals this (...)

This is not possible within the realm of the sign system of algebra, which requires a clear differentiation of referents. The lack of such a differentiation is often accompanied by a common idea of sign-letters as conveying *indeterminacy*. This is made clearer in the following example:

Example 4 (continuation):

1'. Noemi : So, you want to have n plus the next number.
2'. Joe: (*writing the answer*) n plus n?

In this passage, as in many others, the students symbolize 'the next number' as 'n'. And in fact, for many students, all that is unknown is designated by 'n'. Thus, when the teacher went to see the work done by one of our small groups and tried to help them to simplify their symbolic expression by saying, "Then, n+n is equal to ...?", the students promptly answered "n". However, in other instances, as we will see later, it was recognized that a different sign was required to symbolize 'the next number'. As for the relation general/particular in Strategy 2, we see that the particular (through the articulation of numerical facts) offers a powerful tool in the heuristic process. The particular is much more than the realm where to check the correctness of the symbolic general expression.

Strategy 3: In this strategy signs are used in intimate relation to the form and the parts of the figures of the pattern. The relation general/particular plays here a central role in that the referent is clearly emphasized (for instance, having recourse to a geometrical mean). Furthermore the relation general/particular is crucial to the modelling role that the particular will play in the construction of the symbolic algebraic expression. Indeed, the structure of symbolization of the general term of the pattern reflects the structure of the numerical actions of the students. And the particular is often taken as a *metaphor* for the general (for a detailed example of this see Radford 2000). The following example is related to Pattern 3.

Example 6 (continuation):

In this example, the students noticed that the top line always has two more circles than the bottom line. They first built an expression for the number of circles on the bottom line and then added two to it.

1' Anik: n + 1 in brackets plus.
2' Judith: Plus 2.
3' Anik: Plus 1 at first. Look! You do this, then.
4' Jeff: Yes. Then after this it's plus 2.
5' Judith: In brackets.
6' Anik: Yes. Plus 2. [The formula given is: $(n+1)+2=a$]

This example clearly shows how, in this strategy, actions precede symbolization and how the latter is but an expression of the former. The terms "at first" (line 3) and "then after" (line 4) order the temporal numerical sequence of actions at the symbolic level. The syntax of the expression is even dictated by the order of the actions. This is why the student in line 5 says that brackets have to be written. Curiously, in the process of symbolizing, the symbolization of the bottom line of the figure is sometimes left out. In fact, as it appeared during the collective classroom discussion of the activities, some students do not see the need for writing again 'n+1'. They see the formula as indicating a calculation process progressing from bottom to top in a cumulative way. Following with our discussion of the particular/general relation, we see that in this strategy the particular informs in a significant manner the construction of the general expression. This is why the way in which the particular is read somehow anticipates the advent of the general (Radford 1999). Consequently, in this strategy, the particular is not systematically called up to check the validity of the formula. Like in Strategies 1 and 2, in terms of denotation, the sign 'n' is also seen as an *index*. The sign 'n' is pointing to the bottom line of the figure. And, as the first sign 'n' in Example 2 (continuation) line 7', the sign 'a' here is pointing to the result. The whole symbolic expression '$(n+1)+2=a$' can be considered as an *icon* (again, in Peirce's sense) of the concrete figures. Indeed, within its own semiotic space, the symbolic expressions are 'reproducing' the shape of the figures of the pattern.

5. Summary and Conclusion

Our investigation into the students' ways of symbolizing was carried out in terms of the students' production and use of signs as required in activities concerning the algebraic generalization of geometric-numeric patterns. The students' heuristic oriented actions (or strategies) were examined in light of a two dimensional grid (meanings and the general/particular relation). Although these two dimensions cannot account for the whole range of phenomena required to investigate the students' ways of symbolizing (see e.g.

Radford 1999 and 2000 for an analysis of another important dimension related to the students' discourse), the results, nevertheless, shed a new light on the semiotics of generalization. When the three strategies were scanned with the aid of our grid, it appeared that these strategies entail different articulations between the general and the particular. Furthermore, it became apparent that in these strategies the students tend to use signs of a particular sort –*indexical signs*. However, the meaning of signs was different. Indeed, given that indexical signs can signify in a variety of forms, they may bear different indexical meanings. It is true that the fluent algebra user employs indexical signs too. The difference is that the fluent user, in contrast to the novice, can provide the indexical sign with non-indexical meanings. We saw, for instance, how impossible it was to successfully add "n+n" for a group of our students. The difficulty resides in that indexical signs cannot be added. As long as they are still pointing to their objects, one cannot collect them and merge them into a single new symbolic expression. As seen in our discussion of Example 6 (Continuation), the token occurrence of indexical signs unfolded in the realm of an experience sequentially framed in which the signs remained contextually anchored. The algebraic expression is seen as a mnemonic device reflecting the actual course of the flow of calculations. In this sense the algebraic expression functions as a 'performative comment' imbued with the subjectivity of the students' symbolic code which –given its still non-cultural accepted conventional status– requires a 'compiler' to decode it. The possibility to imbue the indexical signs with new meanings stands in need of the creation of new semiotic experiences taking into account these indexical signs. This semiotic experience resides, in part, in that the indexical signs will become the 'objects' of which one thinks, talks and writes. In other terms, they have to become part of a metasemiosis; they have to become part of a new language-game (in Wittgenstein's sense). As for patterns, the question of successfully playing the drama of algebra seems hence to be related, to some important extent, to the possibility of providing the indexical signs with non-indexical meanings.

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STUDENT PERCEPTIONS OF VARIATION IN A SAMPLING SITUATION

Chris Reading
Centre for Cognition Research in Learning
and Teaching,
University of New England, Australia

Mike Shaughnessy
Department of Mathematical Sciences,
Portland State University

Variation is essential to the study of statistics, but may be a neglected topic in school mathematics. Although students may recognize that variation will occur in a sampling situation they have difficulty in discussing reasons for this variation. Three forms of an item were trialed in clinical interviews with students in grades 4 - 12. The interviews probed for students' understanding of centres and spreads which can occur in the resulting sampling distribution for repeated trials of an experiment. This paper contains four case studies of students' reasoning during the interviews.

Introduction

Despite the relative importance of variation to the study of statistics (Wild & Pfannkuch, 1999), there does not appear to have been as much research into students' understandings of measures of dispersion as into their understandings of measures of central tendency. One reason suggested by Shaughnessy (1997) for this deficiency is that research often mirrors the emphasis in curricula material, which tend to focus on measures of central tendency and neglect a careful development of measures of spread. In addition, teachers often avoid teaching spread because they do not wish to introduce the procedurally messy notion of standard deviation.

Given that more students tend to use measures of central tendency than measures of dispersion to reduce and describe sets of data (Reading, 1996), more needs to be uncovered about students' understanding of variability. Despite a lack of questions involving variation in the 1996 National Assessment of Educational Progress (NAEP) in America, one extended response item on sampling that investigated students' reasoning about centres did provide an opportunity for students to discuss spread (Shaughnessy, 1999, p. 2). However, only one student in a sample of 250 responses even raised the issue of spread. This led us to redesign the task to trigger responses that would help find out more about students' understanding of variability.

Research Design

A written task (see Figure 1) similar to the NAEP item, based on sampling from a bowl containing 100 lollies, was piloted with students in schools in Australia and America (Shaughnessy et al., 1999). The results reinforced many phenomena noticed in the NAEP responses but interviews were considered necessary to help explain students' responses. Initially, twelve students were interviewed in Australia, six primary and six secondary. This paper reports on 4 of the interviews, one from each of the grades 4 (Millie), 6 (Jess), 9 (Jane) and 12 (Max).

Students were asked to respond to two different sampling situations: a mixture with 50 red, 30 blue and 20 yellow and another with 70 red, 10 blue and 20 yellow. An actual bowl, containing the correct proportions of wrapped lollies (in some cases coloured plastic bears), was placed in full view. Students were told the lollies were well mixed up. In each case, students were asked how many red lollies could be expected in a handful of 10 lollies. They were then asked to report on the number of reds which would be drawn by six people in a handful of 10 lollies. The lollies were returned to the bowl after each draw and thoroughly re-mixed.

Student Response Form

1A) Suppose we have a bowl with 100 lollies in it. 20 are yellow, 50 are red, and 30 are blue.
Suppose you pick out 10 lollies. How many reds do you expect to get? __ Would this happen every time?
1B) Altogether six of you do this experiment.
What do you think is likely to occur for the numbers of red lollies that are written down? Please write them here. _____, _____, _____, _____, _____, _____
Why are likely numbers for the reds?
1C) Look at these possibilities that some students have written down for the numbers they thought likely.
Which one of these lists do you think best describes what might happen? Circle it.
a) 5,9,7,6,8,7
b) 3,7,5,8,5,4
c) 5,5,5,5,5,5
d) 2,3,4,3,4,4
e) 7,7,7,7,7,7
f) 3,0,9,2,8,5
g) 10,10,10,10,10,10
Why do you think the list you chose best describes what might happen?

1D) Suppose that 6 students did the experiment--pulled out ten lollies from this bowl, wrote down the number of reds, put them back, mixed them up.
What do you think the numbers will most likely go from? From ____ (low) to ____ (high) number of reds.
Why do you think this?
**(After doing the experiment) Would you make any changes to your answers in 1B-1D? If so, write the changes here.
1E) Suppose that 6 students each pulled out 50 lollies from this bowl, wrote down the number of reds, put them back, mixed them up.
What do you think the numbers will most likely go from this time?
From ______ (low) to ______ (high) number of reds.
Why do you think this?
1F) Suppose that 40 students pulled out 10 lollies from the bowl, wrote down the number of reds, put them back, mixed them up. Can you describe what the numbers would be, what they'd look like?
Why do you think this?
.... These questions were then repeated for the 70 red situation.

Figure 1 - Student Response Form (Condensed)

Responses were sought in three different forms, LIST (write the number of reds in each handful), CHOICE (choose one of seven different multiple choice options) and RANGE (give the lowest and highest number of reds). Students were also asked to explain their responses and then given the chance to alter any of their responses after having actually drawn six samples of 10 from the bowl. The extended questions, parts E and F of the response form (Figure 1), were only presented to Year 12.

Responses were coded on two dimensions, centre and spread. Centres were coded as LOW, MEAN-CENTRED or HIGH depending on the central tendency. Spreads were coded as NARROW, REASONABLE or WIDE according to the dispersion. This coding (Shaughnessy et al., 1999) was based on information obtained from student responses to a written version of the task and was used to build a profile of each student's perception of the sampling situation. For example, a response 5 7 9 8 6 5 would be coded as HIGH-REASONABLE for the 50-red mixture, since the centre of the student's response was high compared to the expected mean (5 in this case) and the range of the numbers of reds pulled was neither really narrow nor really wide. Similarly, 5 5 5 6 5 5 would be coded MEAN-CENTRED but WIDE and 1 3 5 7 9 10 would be coded MEAN-CENTRED and WIDE.

The interviews were designed to compare students' responses across the three versions of the tasks, and to investigate students' perceptions of variation. More specifically, the following questions we investigated: Would one form of the question LIST, CHOICE or RANGE give most information about students' conceptions of variation? How do students' describe variation in a sampling situation? What reasons will students give for their responses? Does the proportion of the colours influence the response? How does actually conducting the sampling experiment alter the responses? Are the students' responses consistent across versions of the task? What is the students' overall perception of the sampling distribution for this task? Can they attend to a range of likely outcomes, as opposed to focusing on single outcomes, that is, will they consider spread as well as centre in their responses? The four case studies will be considered, and then we will summarize student thinking on these questions. Directly quoted comments from students are presented in italics.

Millie

Millie suggested one handful, for the 50-red situation, would yield 3 or 4 red and appreciated that it should be different each time. Her LIST (5 2 3 6 4 3) and CHOICE (2 3 4 3 4 4) responses were also low. Her explanation *lots of other colours as well*, suggests that she was not considering that there were 50 red, only that she would expect to get some of each of the 3 colours. It is possible that Millie was thinking according to an 'equal probability' bias, or that she thought each colour would come out with the same 'fair' chance. Millie did not want to choose the option where all numbers were the same, indicating an appreciation of the variation of outcomes. Millie gave responses with reasonable spread. Her RANGE (3 to 6) suggested a mean-centred response but when details of the spread are considered across the three forms of the response she was responding quite low. Millie decided not to change her responses after doing the experiment (1 3 5 4 6 4) because, although the experiment yielded a 1, she decided against including a 1 as she *may not get a 1 next time*.

Although Millie still chose 4 as the number of reds for the 70-red situation, she did LIST (3 6 5 4 4 7) some higher numbers, because *there were more reds than before*. She was still not able to focus in on the number of reds, just comparing it to the previous 50-red situation. Her CHOICE (3 7 5 8 5 4) and RANGE (3 to 7) responses were reasonable in spread but too low. As before, Millie did not want all numbers to be the same. After completing the experiment (7 5 5 7 7 7) Millie changed one 4 to a 6 and the other to a 5. The experiment has suggested to her that her response was too low, however she still was not concerned about the 3 she gave.

Generally speaking Millie's responses are too low. Millie did not explicitly focus on the number of reds in the mixture, and never considered the relative ratios of the colours. Although her ranges are reasonable she appears to consider that variation will occur evenly over her preferred range. Millie generally had difficulty in explaining why she chose the responses that she did.

Jess

Jess expected more red as one can *see more red* and despite claiming that the exact number of reds did not matter, she predicted 5 in one handful for the 50-red situation.

Her appreciation for variation was evident when she said it would not be the same every time. Jess gave mean-centred responses for CHOICE (3 7 5 8 5 4) and RANGE (3 to 8) but her LIST (5 4 6 3 4 8) was low. All three responses were reasonable in width. Jess chose her CHOICE response because it was *more spread out*, meaning that she didn't want the same number each time, rather than wanting to include all of the numbers (as one would expect 'more spread out' to mean). In fact, Jess stated that the size of the number didn't matter. Then, when explaining that extremes, such as 1 or 2, were not possible her reason was that *there are 50 red*. After performing the experiment (5 6 4 5 8 3) Jess chose to change the 2 to an 8 in her LIST, making the response mean-centred rather than low. Her reason was that they were *all mixed up*, but she did go on to say that it was hard to explain.

Jess suggested that 9 may come out in the handful for the 70-red situation, although she did suggest 7, 8 or 6 for next time. Jess gave mean-centred and reasonable width responses for the LIST (9 7 5 6 8 9), CHOICE (5 9 7 6 8 7) and RANGE (5 to 9). Her explanations basically wanted higher numbers and, although mentioning the 70 red, she just said that numbers should be picked *in the higher half of 5* (meaning 5 to 10). Jess did not want to perform the experiment this time.

Jess had a better feel for the sampling distribution than Millie, having both mean-centred and reasonable spread responses in both sampling tasks. However, Jess was still not able to give explicit reasons for her responses in terms of the colour proportions and any variation she indicated was fairly uniform in manner.

Jane

Jane suggested 5 in a handful for the 50-red situation and then said that 5 could happen again. However, she also stated that it could be something else, even allowing that *probably could get all reds*. Although her LIST (5 7 6 2 4 3) was mean-centred, her RANGE (4 to 10) and CHOICE (5 9 7 6 8 7) responses were high. All three had reasonable spread. She could not clearly explain why she gave the LIST and RANGE responses but made the CHOICE response because she wanted some *mixed up* and *two doubles*. Jane felt that someone might get doubles. This indicates that Jane was not just comfortable with the idea of the same number appearing more than once, but seemed to think that it was necessary. The response *probably not enough for all six of them to pull out 5 red*, was confusing as the actual numbers Jane gave were higher which would require even more total red. She may have forgotten that the lollies were replaced each time. Jane's responses are representative of many students (on the written version) who responded with the total number of reds drawn for all six handfuls, rather than one handful at a time. After performing the experiment (7 5 6 6 3 6) Jane made no changes and justified her actions by matching up, number for number, the experimental results with her responses.

Jane suggested that 10 would be drawn out for the 70-red situation because *there's heaps of reds in there* and in fact when writing her LIST (10 7 8 10 9 10) she included 10 many times. The RANGE (9 to 10) also focused on the very high values.

However, with the CHOICE (5 9 7 6 8 7) Jane gave a more mean-centred response, even though it was not consistent with the other responses. She had avoided the all 10 choice though because she felt that the numbers could not all be the same. Having completed the experiment (8 9 6 6 5 8) Jane changed the RANGE to 5 to 10.

Jane seems to fixate on the large number of reds in the bowl, especially in the second version of the task. She has a tendency to give high, though reasonable spread, responses. Choosing 5 for the 50-red situation suggests that she had some appreciation of the 50% red mix even if she couldn't explain her response. However, her ratio concept is not solid, since the fact that she estimates 10 for the 70 red situation suggests that she views this as a 'more than 50%' situation rather than taking note of the 70% mix. Jane also appears to better represent the spread of the values for the 50 rather than the 70 red situation, suggesting again that she is more comfortable with the 50% mix.

Max

Max suggested getting about half red in the 50-red situation but gave the complicated ratio of 2.5 to 1 as the ratio to yellow when explaining his response. However, he kept his options open by adding *but you never know you could have a bit of luck one time*. His LIST (3 4 5 5 6 4) together with his reason *I tend to think that it could be averaged around half each time - so around there somewhere, but I think it would go over a number occasionally just because of luck basically* indicated that he both appreciated, and could explain, the influence of half the lollies being red. The RANGE (3 to 7) and its justification, *the average, around in the middle, about half way* also demonstrated this. Both responses are mean-centred and reasonable width and indicate a good appreciation of the 50% mix and variation amongst the outcomes.

Initially, the CHOICE (5 5 5 5 5 5) response was based on *I think there would be more of an average, 5 each time* but then Max decided to change to 3 7 5 8 5 4 because he felt that *the chances of getting 5 every time are not high but the average would need to be around 5*. Max is torn between wanting the long term average to be 5 and also wanting to demonstrate variability. This demonstrates interference from probability instruction, in which the focus is on the chance of single occurrences of an outcome (probability of an event or most likely event), rather than on a reasonable range of outcomes (the notion of a distribution). After completing the experiment (4 5 3 4 6 5), Max was more convinced that his CHOICE should be changed and wanted something more spread than all 5s, although he worried that 8 was too high. Max was also presented students pulling out 50 lollies in each handful. He had trouble giving a RANGE response, changing from 12 to 17 to 17 to 20 and then 20 to 26. At first guessing, he eventually arrived at 20 to 26 after realizing that half of the lollies were being pulled out. However, Max was not able to articulate that half of 50 was 25.

When asked how many in one handful for the 70-red situation, Max replied *at least 6 but chance it could go to 10*, already indicating that variation was possible when asked for a single response. However, his reason only mentioned *many red*, not the proportion. He did include the need to be mixed really well, though. All three responses, LIST (5 8 7 6 6 8), CHOICE (5 9 7 6 8 7) and RANGE (5 to 9) were mean-centred and reasonable in width. However, his reasons generally reflected a need for the numbers to be at least 5. It was not clear why 5 was chosen. After the experiment (8 4 6 7 7 9) Max considered changing a 5 to a 4 but then decided that the 4 was *just luck anyway* and if the experiment was repeated it may be a 5 and then he would have to change again. Finally, he decided that if there was a chance of getting 4 then it should be included. Now, when presented with selections of 50 lollies Max gave 25 to 40 as the RANGE with the reason that *definitely have to go over half because you have over a half of the reds*. Then, when considering that there are in fact 70% red decided 50 red was possible.

When confronted with the extended problem of 40 students pulling out 10 lollies each, Max described the results as *follow the same pattern*. He decided it would be more spread, around 4 to 9 with the same number of 5, 6, 7, 8 and 9 appearing and the *average around 7 or something*. He even chose to allow for the fact that he may get 2 or 3. Interestingly, Max allowed for the fact that there could be more variation but was still keen to allow equal occurrence of numbers.

Unlike the younger students, Max was able to articulate the effect that the proportion of reds has on the outcomes and he demonstrated a good appreciation of the variation. On the other hand, Max exhibited some interference with his probability concepts in this task. Max at first chose 5 5 5 5 5 5 as the result for the sampling task, and also said that it would approach 5 as an average in the long run. These are answers to different questions than the ones being asking in the task.

Discussion

There appears to be a steady growth throughout grades 4 to 12 in the ability of students to explicitly describe the sampling situation on the centring scale, with language that starts referring to the 'number of red' from Jess (Grade 6) to some explicit use of ratio by Jane (Grade 9) and finally to Max's (Grade 12) explicit use of the 50% and 70% ratios in the two tasks. However, on the spread scale performance is somewhat oscillatory, and responses may not be consistent across versions of the task (see Jane, for example). Particularly obvious was the Year 12 student's initial tendency to give narrow responses. As mentioned above, this could be due to some interference fom instruction in probability, and to lack of instruction in sampling distributions. This tendency has been confirmed in the analysis of the written version of the tasks (Shaughnessy et al., 1999). Max did appear to 'learn' as the interview proceeded, and his later responses indicated a more 'reasonable' spread. Students appear to be more capable of giving explicit reasons for their responses when describing the central tendency of the results of the sampling situation than they are for describing their reasons for dispersion. Mostly, students would say 'well, they won't all be the same'. Except for Max, there was no discussion that the results should cluster around the expected mean number of reds in the mixture.

These students tended to be consistent in their responses across the three forms of the question. However, the LIST form of the question appears to give more information about variability in the sampling situation than the CHOICE or RANGE versions. Generating their own set of outcomes, allows students to demonstrate more about their implicit concept of a sampling distribution for a small number of trials, than when possibilities are suggested to them (as in the CHOICE version). Generally, these students seemed to prefer the RANGE version of the task, as they said it was easier. However, this simplicity meant that it did not give as much information about students' expectations of mean-centredness and spread, as either of the other two forms. The CHOICE form of the question was not so useful as a measure for two reasons. First, it did not allow enough options for those who had a tendency to choose wide. This could be remedied by adding more options but then the question would become unmanageable. Second, students would not have been able to demonstrate their preference for more even spread of the possibilities over their preferred range if they had been restricted to just answering a CHOICE question.

It may be hard for students to describe variation with only six handfuls. For example, if the student decided that a suitable range for the 50-red situation is 3 to 7, then he or she tended to try and cover the whole range when giving the six numbers to show that each is possible. This left little scope for indicating variation, usually giving the impression that the student expects an even distribution of outcomes. Hence the 40-student question, added for Max, may give more information. However, even with the scope of 40 numbers, Max still seemed to expect an even distribution. The three younger students all justified the range chosen by giving reasons why extreme values should not be included. It was only Max who discussed average when justifying the range given, wanting numbers *around in the middle*. In a revised version of the sampling task now being used with secondary students, we ask them to imagine repeating the task 100 times, and then to draw a histogram for the frequency of the number of reds pulled (labeled axes given). It would perhaps be even better to then include a computer simulation of the sampling task, so that several distributions of 100 trials could be quickly generated, and ask the students what they think, and if they would change their own graphs.

When indicating variation students do not use specific words such as 'vary', 'deviate', 'fluctuate' or 'variation'. This observation is consistent which the results observed for the written responses (Shaughnessy et al., 1999). Millie and Max, however, make use of the expression 'more spread'. Millie meant she just wanted different numbers, that is, not the same number every time. Similarly, Max gave this as an explanation of why he no longer thought his first choice of the all 5s option was suitable. Although students do give some indication of the variation that they expect by choosing particular numbers, there is very little 'discussion' of that variation which takes place.

The students' justifications for responses are interesting in terms of the reasons they gave for the range. For low range estimates students often indicated a preoccupation with totals, while for high range predictions students were usually concerned with the large number of reds rather than the proportion of reds. When the Year 12 student, Max, chose the all 5 option sampling issues may have become confounded with calculating probabilities.

Justifications also suggest that some students try to 'explain' the 'unexplainable' by finding reasons for the 'variation' within the set of numbers given. If the experimental results are not self-confirming with their predictions, students try to make them so with their explanations, which might include references to 'variables' such as the size of the hand, how well the lollies are mixed and the position selected from in the bowl. There also appears to be a need to 'be right' in predicting the results of the sampling. Students sometimes 'think hard' when making their choice on the LIST version of the task.

The proportion of colours in the bowl does appear to effect a student's ability to predict the outcomes. With so many student experiences involving the notion of a half it is not surprising that the younger students cope better with the 50% mix than the 70% mix. Even though mean-centred, reasonable-width responses can be given, and often explained, for the situation where half the lollies are red, as soon as the proportion becomes unbalanced, with 70% red, students only seem able to deal with situation as being 'greater than 5'. Also, it appears to be more difficult for students to justify their responses with the 70% mix.

Conclusion

On existing evidence it appears that the LIST form for the question is the most useful for getting students to describe the sampling situation. However, it may be that the situations with many more people, such as 40, for the experiment will most likely encourage students to engage in a discussion of possibilities which will include consideration of the variation. Students improve with age in their ability to describe the sampling situation but are not able to articulate well the reasons for their responses. The uniformity of the variation that they expect suggests that students may need to experience more such sampling situations to better appreciate the possible variation. Also, more experiences need to be presented to students which involve proportions other than 50% mixes.

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CAPSTONE COURSES IN PROBLEM SOLVING FOR PROSPECTIVE SECONDARY TEACHERS: EFFECTS ON BELIEFS AND TEACHING PRACTICES*

CHERYL RODDICK
JOANNE ROSSI BECKER
BARBARA J. PENCE
SAN JOSE STATE UNIVERSITY, CA, USA

Abstract

Two innovative courses in problem solving were taught in 1998-99 for prospective secondary school teachers. A multifaceted approach was undertaken to evaluate the results of the courses on students' problem solving, problem posing, modeling, and beliefs about the role of problem solving in teaching mathematics. This paper reports on the beliefs and how they have changed of three students who completed both courses. Data sources include journal entries and reflections on problem solving, and in-depth interviews four months after completion of the courses. The three students chosen to highlight in this paper fell on a continuum ranging from not much discernible implementation of problem solving to substantial integration of problem solving into one's teaching.

Focus of the Paper

This study focused on evaluating the effects of two problem-solving courses on the beliefs about problem solving of prospective and practicing secondary mathematics teachers. It is part of a larger study of the influence of the courses on problem-solving abilities, beliefs about problem solving, and problem-solving practices in the classroom.

Background

We briefly discuss here research on beliefs relative to problem solving of both students and teachers, as the courses described below were designed for pre-service teachers, many of whom would be in a classroom the next year. Views commonly held by students on the nature of problem solving include the beliefs that there is only one right answer to a problem, and only one correct method of solution (Schoenfeld, 1992). In case studies of teachers' beliefs and how they affect classroom instruction, Thompson (1985) presents the case of Kay, whose beliefs about pedagogy included two especially relevant ones: that the teacher should create an open and informal classroom environment to insure students' freedom to explore their own ideas; and, that the teacher should encourage students to guess, conjecture and reason things on their own. Kay's classroom behavior was consistent with her beliefs and was supportive of the development of students' problem-solving abilities.

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McLeod has perhaps written the most about affective variables and problem solving (e.g. McLeod, 1993). In that paper, he discusses attributes of teachers who were successful in dealing with affect in teaching problem solving, such as presenting themselves as problem solvers, making frequent use of cooperative groups, and valuing problem solving processes. How does one develop beliefs such as Kay's and attributes such as those McLeod highlights? As Lester (1994) states in a review of problem solving, more research is needed on problem solving instruction to try to answer these questions. By modeling the teacher behaviors identified by McLeod (1993) and Thompson (1985) in the teaching of the classes, and explicitly discussing these as valuable teacher behaviors, we attempted to develop requisite beliefs that might translate into instruction supportive of problem solving. This study begins to answer the question of whether this approach was successful.

Description of Coursework

The two graduate courses, Math 201A and 201B [Secondary School Mathematics] were completely redesigned in the 1998-99 academic year to adhere to new California state standards for those seeking a secondary school credential in mathematics. The second and third authors spent much of the summer of 1998 planning the courses, and team taught parts A and B in Fall and Spring respectively. The broad goals for both courses were to:

- Explore problems from strands of number theory, algebra & geometry;
- Investigate modeling in mathematics;
- Improve students' problem solving abilities;
- Apply inductive and deductive reasoning;
- Learn ways to assess problem solving;
- Enhance students' understanding of equity issues in teaching mathematics;
- Broaden students' views of problem solving and of mathematics more generally; and,
- Influence teaching practice toward implementation of the National Council of Teachers of Mathematics standards (NCTM, 1998).

In part A, the course focused specifically on problem posing and modeling (Brown, 1996; Dossey, 1996). Students spent significant time on topics such as: what is a problem; finite differences; examination of problem solving in traditional and innovative curricula; equity issues in problem solving and its assessment; assessment of problem solving; and use of technology. In part B, the students used a model for reflecting on one's problem solving (Mason, Burton, & Stacey, 1985), and concentrated on specializing, generalizing, and justifying their work. Both courses included substantial in-class time working in groups on problems and giving presentations and justifications to the class. Students completed two group projects in each course, as well as more traditional homework and exams (although these were group as well).

Methodology

The majority of the students in the two courses are seeking a secondary credential in mathematics. However, the courses are also required in a master's degree program for secondary teachers. In the 1998-99 academic year, there were 24 students in the first course and 20 in the second. Although the first problem solving course was not a prerequisite for the second, there were eight students who took both courses. Six agreed to participate in this study. This sample included four females and two males, one of whom was Asian. While taking the two courses, two study participants were also full time teachers on emergency credentials, one was still an undergraduate working on a degree in mathematics, and the rest were graduate students. In the final phase of data collection, one additional participant completed her student teaching. In this report, we will focus on these three subjects who were teaching mathematics during any phase of data collection, with major emphasis on one case.

Instruments used for this study included both written and interview data. An initial journal write was conducted in the first week of the fall class and a final reflection was given during the last week of the spring class. The first task asked the students:

> If problem solving were a building, what kind of building would it be? Why? (Gibson, 1994).

The final reflection for the spring contained three items but the one included in this analysis stated:

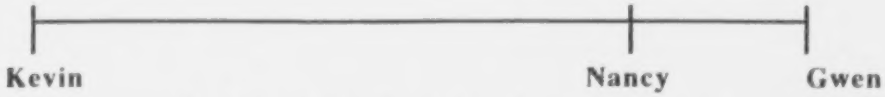
> This comment came up during one of our class discussions. Please react. "Before a teacher engages secondary school students in problem solving, those students need to have mastered the required mathematical content first."

Four months after completing the two courses, the participants were interviewed individually. The interview had five sections, including: demographics; the nature of problem solving; problem-solving strategies; the role of problem solving in curriculum and instruction; and assessment of problem solving. A twelve-item protocol was developed to structure the one hour interview.

Protocol development for the interview involved all three of the authors. In order to encourage open discussion in the interview, the interviewer, the first author, had not participated in teaching the courses. Interviews were audio-taped and transcribed by the interviewer. Data from the interviews were analyzed in two ways: section by section across the six students to discern patterns in responses; and, a holistic examination of each student's interview across the sections. The three different data points, which span 12 months, were examined for evidence of change in the students' beliefs about problem-solving instruction.

Results

The three participants in this study, here called Kevin, Nancy, and Gwen, seemed to fall on a continuum when examined from the perspective of implementation of problem solving in their teaching. As shown in the figure below, Kevin fell on one end of the continuum indicating insubstantial problem-solving implementation, while Nancy and Gwen were classified towards the opposite end, indicating substantial accomplishment. Gwen was the subject who showed most growth in her thinking about and implementation of problem solving in her instruction.



However, we should note that Kevin has changed in his perspectives about problem solving and realizes he should be incorporating more in his teaching, but he faces pressures from state- and district-mandated assessment and the need to cover curriculum. Nancy makes sure she does the one section in each chapter of her textbook that focuses on problem solving; but she thinks more should be in the curriculum, so she has designed some projects that incorporate problem solving. Also, Nancy is making use of rubric scoring (a holistic approach) in assessing projects and homework. Because the problems she assigns can be solved in multiple ways, Nancy feels that she has to assess the process, not just the solution.

Due to space limitations, this paper will concentrate its discussion with the case of Gwen. Gwen was chosen not because she is representative of the sample, but because she exemplifies, from self-report data, the type of mathematics instruction incorporating problem solving that the courses were designed to stimulate. Gwen is a 46-year old female who, after raising a family, returned to school and received her B.S in mathematics in 1998. She is completing her teaching credential while teaching high school mathematics full time. Gwen was a first-year teacher when she was enrolled in the two problem-solving courses, and had begun her second year of teaching at the time of the interview. Her responses will be discussed around the four major sections of the interview.

The Nature of Problem Solving

At the beginning of the first problem-solving class, the participants were asked the question, If problem solving were a building, what building would it be? Gwen's response indicated a view of problem solving that was algorithmic and maze-like:

> If problem solving were a building, it would be a government building where you are sent from department to department in order to complete your request. In problem solving, there are many steps and various strategies which can often feel like a bueracratic (sic) maze.

This response serves as baseline information concerning Gwen's view of problem solving at the onset of the course. In the interview, which took place after the two problem-solving courses were completed, she was asked to describe problem solving in mathematics. She described it as something that was "not always clear cut," an opportunity to use mathematics to make sense of a situation or come to a conclusion.

> It's kind of like being a detective, investigating and looking and using what you've got.

Her example sheds additional light on her beliefs about what constitutes a "problem." She proposes the game of NIM as an example of problem solving.* She notes that the problem is not clearly defined; there is investigation involved in arriving at a winning strategy, and that is not something they can just plug into a formula to achieve.

To have Gwen reflect on how her view of problem solving had evolved over the course of the two problem-solving classes, we asked her how she thought she would have described problem solving two years ago. According to her response, she believes that her understanding of problem solving has been enriched:

> It would be like more application or somehow prove something with like a proof possibly. Or I don't know if it would be so different , but I don't think I would have allowed it to be such an open-ended (question). I would have felt that it had to be more structured and more rigid. And maybe only one solution to this problem.

It is interesting to note that this description of a problem as "structured" and "rigid" complements her view of problem solving as a government building, which represented her beliefs *before* taking the problem-solving courses. This view contrasts sharply with her view *after* the courses, when her use of the detective metaphor alludes to non-sequential approaches to solving a problem.

Problem-solving Strategies

Gwen was then asked to describe the different strategies she uses when approaching an unfamiliar problem. She mentioned some common problem-solving strategies, such as trying to relate it to a similar problem, solving a simpler problem, generating tables, and looking for patterns. The strategy of looking for patterns was one that she attributed to learning in the two problem-solving courses. In fact, before the courses, she would not even have considered it to be a mathematical approach.

> It's kind of like guess and check. That's not very mathematical. But it was in (201A) that I saw that it's very mathematical. It's a basis for a

---

* The game of NIM involves two players and a given number of sticks. Each player can remove one to *n* sticks per turn. The winner is the one who removes the last stick.

> lot of the wonderful discoveries of mathematics we have. First they saw the patterns and then generalized them.

When asked specifically about what she did when "stuck," (Mason, Burton & Stacey, 1985), Gwen put a lot of emphasis on talking with other people.

> I go to my group, get other ideas from other people. . . .And what I do learn working in my group was to be more receptive to people's ideas that seemed (ridiculous), that's not going to work, then three hours later we found in fact that was going to work. . . . Today I would hope I would respond, why do you think, what makes you think that's going to work? Why do you think that's applicable here? Have them say more about it before I dismiss it as nonsense. Because before I was just like, I don't see it, so it can't be. Now I think even if I thought it was, I would hope that I would at least say can you tell me, say more about this? To get an idea.

When asked about the role reflection plays in her problem-solving process, she said she usually does not stop to reflect until she gets stuck. It is at that point that she will stop to think about what she has done and why she is no longer making progress. Then, once she has finished a problem, reflection is a way for her to "solidify" the information, and to add her new understanding to existing understanding.

The Role of Problem Solving in Curriculum and Instruction

The next part of the interview focused on problem solving as it is implemented in the 7-12 mathematics curriculum. In Gwen's opinion, there is not a lot of problem solving incorporated into the curriculum, and, while the text that she uses integrates problem solving throughout, the problems are not as open ended as she would like. She notes that there is a definite direction inherent in most of the book problems. It is precisely this facet of problem solving in schools that she would like to broaden. In her opinion, problem solving should be integrated throughout the curriculum and it should be used to encourage student exploration of different approaches and solutions.

> Students need to get out of the box that they put themselves in or somebody put them in where there's only one way to do it.. . . It's rare that students come with the love of 'just let's see what happens if we do this. Let's talk about that; what about that?'"

She speaks of her desire for her students to "engage joyfully in an exploration of a problem." This is one of her goals for her classes. During her second year of teaching, she reports that she has spent more time in problem-solving explorations, and she plans to spend more time moving her students towards more open-ended questions. She gave two examples of the types of problems she has used with her students: How many squares are on an 8x8 checkerboard? and, The Game of NIM. These problems are rich enough to allow students to explore different avenues, search for patterns, attempt a generalization, and extend the problem given. Gwen was adamant in her goal of wanting students to enjoy themselves throughout the process of problem solving.

Assessment

Because Gwen is regularly incorporating problem solving in her instruction, she had strong ideas about how to assess students' work. She used a five-point rubric but was lenient in her grading; her goals tended more toward encouraging student thinking and enjoyment of the problem solving process.

> They are so focused on getting the right answer they don't enjoy the journey.

Her rubric was designed to assess different approaches to solving the problem, the use of strategies such as looking for a pattern, and the ability to generalize and find extensions.

> But what I look for is how many different approaches. I asked them to show me ALL their work, whether they succeeded or not. I wanted to see what they did, what their thought was, and if they tried.

Summary

Although Gwen was a mature learner, she obviously has experienced considerable growth in her views of problem solving and its role in instruction. Based on self-report, she seems to have genuinely incorporated her learning from the two problem-solving courses into her own classroom practices. It seems that she seeks to recreate in her own classes the atmosphere she felt in the problem solving courses. In particular, Gwen reported that she exemplifies some of the positive beliefs and attributes discussed in Thompson (1985) and McLeod (1993). She encourages exploration and uses cooperative groups for her problem solving work, and she obviously values students' processes through her assessment.

The progress Gwen made as an individual learner was also of interest. She entered the courses thinking she was not very good at mathematics, and felt

> "it was such a gift to get the info that you don't have to have all the answers. . . . I don't care if I'm not a math whiz. It's fun for me now! It was a gift that I got out of 201A and B, that it can be fun."

Conclusion

Although this report represents a case study of one person, it demonstrates the changes that can occur in beliefs and instruction as a result of an intensive, year-long course that immerses students in being reflective problem solvers themselves. A followup study involving classroom observations of course participants would certainly complement this self-report study.

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EXPERIENCING THE NECESSITY OF A MATHEMATICAL STATEMENT

Catherine SACKUR, Jean-Philippe DROUHARD, Maryse MAUREL
GECO - IREM, Université de Nice, France
Teresa ASSUDE
DIDIREM, IUFM de Versailles, France

A teaching device tends to permit students to experience the necessity of mathematical statements (here in spatial geometry). We emphasize the role of the confrontation between students to have them come up against what could be called "the mathematical reality". We describe four steps in such a teaching, going from a personal work of students to a sequence when the whole class collects through work in small groups of four students. In front of the whole class, the teacher plays an important role to bring into light the learned knowledge, its necessity in mathematics and the way some students experienced this necessity.

Introduction :

In this presentation, we will be interested with the teaching of

the character of necessity of mathematical knowledge

Let us observe the statement:

« In a parallelogram, if the sides are equal, the diagonals are perpendicular ».

All mathematicians know that the content of this statement cannot be different. The truth of this statement comes from the mathematics themselves, from the axioms and from the rules of demonstration in mathematics. It comes, neither from observation of nature, nor from any arbitrary choice of the mathematicians. This is the case with the mathematical knowledge which is taught at school or in the first years of university. From this point of view, mathematical statements differ from many other statements such as: « Mount Fuji is 3776 meters high ».

We observe that many students don't know this characteristic of mathematical knowledge. They quite often think that things could be different if we decide so, especially in Algebra where the rules of computation seem arbitrary to them; thus their knowledge is not coherent. We are then interested in teaching this characteristic of « necessity » in mathematics and we believe that it cannot be taught directly through a discourse of the teacher.

We shall describe here a school-room situation in which the necessity of a mathematical statement is brought into light and is then institutionnalized by the teacher as the same time as the knowledge itself.

In the first part of our presentation we shall present the theoretical frame of our work, then describe the teaching device and in the third part we shall analyse the activity of the students and interpret it.

The knowledge on which we have been working is the following:

in an Euclidian space, the equation $ax + by = c$ *must be the equation of a plane.*

I. Theoretical Background

Mathematicians are well aware that mathematical truth has a character of necessity, in the meaning we explained in the introduction (we speak here of properties, theorems etc., obviously not of definitions or axioms).

Few students however, show such awareness. We even interviewed a 16 years old student, average level, who complained having been bored with algebra because "it was made of a stack of unrelated rules"! Most commonly, teachers think that *demonstrations* do convince students that mathematical properties are necessarily true. It is clear indeed that a *given* demonstration may establish that a *given* property is necessary. However, it is the case just with the few students who are *already* aware that - in general - demonstrations have something to do with necessity. We tend to say that to demonstrate may *reinforce* the conviction that in mathematics, truth is necessary, but cannot *initiate* this conviction.

One can say that something is necessary, only if one *may* think that things *could* be different, but *are* not. In the realm of physics for instance, two isolated magnets will be said *necessarily* attracting one another, insofar as one can imagine non-attracting magnets, meanwhile it is not possible that they do not attract one another.

Now, what could be a mathematical equivalent of the physical world, where the subject can imagine that things could differ, and experiment how things behave? This point is related to the question of the "milieu" in G. Brousseau's Theory of Situations (1997) or to "real" world in constructivist theories (Piaget, in his studies on necessity (1981, 1983)) even though we don't know this "real" world, (Von Glasersfeld 1996).

We claim that the interaction with 'Others' (which may be not convinced, disagree, have another point of view etc., Drouhard, 1997) permits both to make the different possibilities thinkable and to experience a conflict (with the Others' disagreement). This experience is constitutive of the experience of the necessity (Drouhard, Sackur, Maurel, Paquelier & Assude, 1999). We meet here authors like Ernest (1997) in the importance given to the social interaction in the subject's construction of mathematics. The "mathematical discussions" as described by Bartolini-Bussi (1991) are a good example of teaching devices taking into account this importance.

We are then interested in teaching situations centered on necessity, in which peer interactions play an important role.

Wittgenstein helped us to figure out that social and language aspects are literally essential to understand the very nature of the necessity of mathematical knowledge. We found in Wittgenstein (1978) a subtle characterisation of necessity, related to resistance, although apparently paradoxical: the idea (expressed here in a very sketchy way) that mathematical objects resist us to the very extent that we *want* them to resist, and not because of their physical nature as walls do. But, what is the origin of this willpower? It seems that, for Wittgenstein, mathematicians need objects that resist them because *doing mathematics* is precisely *working on such* resisting *objects*. Surely they *could* work with 'weaker' mental objects but then, what they would do, could no longer be called "mathematics"(Drouhard & al 1999).

As we said before, it is not sufficient for a teacher to claim that mathematical statements are necessary to convince students of this property. We think that this conviction has to be acquiered through the reconstruction by the student of the meaning of the statement. Following Husserl (1936) and the phenomenological approach, this reconstruction of meaning is then elaborated into an experience that the student can live again, if needed, either in the same context or as a general attitude.

The emphasis given to peer interaction must not lower the teacher's role. Not only is he supposed to set up a convenient 'milieu' for the learning of the (necessary) knowledge, but he also plays an essential role in helping students to be conscious of this character of necessity (during the phase of "institutionalization" in the terms of the theory of didactical situations).

II. The Work in the Class

1. purpose of the teaching

Three groups of forty students from the first year of university (age 19 and20) worked following this device. Their knowledge about equation of planes and lines was the following:

They had knowned for a long time that in the plane, $ax + by = c$ is the equation of a straight line. During the last year of high school they learn that in the space the equation of a plane is $ax + by + cz = d$, and that a straight line is defined by a set a two equations as it is the intersection of two planes. Of course when one variable is missing in the equation, although there are taught that they are working in a space, the old knowledge comes back and they say that $ax + by = c$ is the equation of a straight line. The purpose of this work is then to make them work on their old knowledge, correct it, and, at the same time, teach them that the correct knowledge is necessary.

2. schedule

There were five phases in the device:

- first and second phases (the same session): 20 minutes of personal work, followed by one hour of work in small groups of four students,
- third phase (second session, one or two days later): synthesis in each large group; report of the small groups, agreement on the result, institutionnalisation by the teacher,
- fourth phase (one week later): work on the link between planes in $\mathbb{R}^3$ and linear systems. Work on change of settings.
- fifth phase (one week later): the three large groups together in a « regular » class, course on equations of lines and planes in $\mathbb{R}^3$.

3. a-priori analysis

We will explain here the role of the three first phases of the work. During the **first phase**, the students are working alone; this is the time when their own knowledge is activated, when they make up their mind about the answer to the problem, using what they know and when some of them produce the expected error.

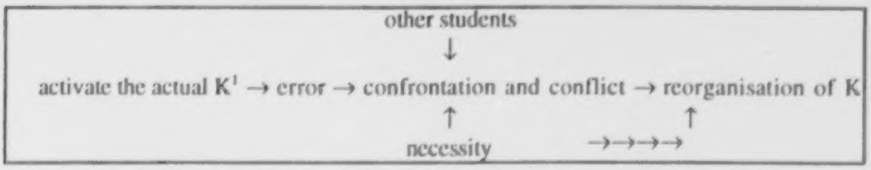
During the **second phase**, when they are working together in small groups of four students, they have time to:

- confront their opinion with the opinions of *others* and choose an answer; here one finds the role of the conflict between the students.

- become sure of the validity of this answer through the discussion with others. An important point of the device is that the teacher has asked that at the end of the discussion *any of them* should to be able to give the report of their work. So they have to agree and be convinced.
- find an agreement based on mathematics. In a way, one could say that the mathematics « decide » what is the correct answer to the problem. At that moment the students experience the « necessity » of the statement. In this case they experienced the necessity that the given equations were equations of planes and surfaces and not of lines and curves.

In the **third phase**, the synthesis takes place. The large group gathers, and each small group tells about its work; it is the moment when the teacher can institutionnalize this experience of necessity as it has been experienced in some of the groups. It is clear that all the groups do not reach the same state of knowledge, and do not experience the necessity in the same way. The teacher can take advantage of the experience of some of them to make clear the way which led to this experience. Together with the mathematical knowledge, this is what is brought to light by the teacher. Following our theoretical elaboration, this characteristic of the knowledge cannot be separated from it. A mathematical knowledge is « necessary » otherwise it is not mathematical.

One can sum up the film of what takes place in the following diagram:


other students
activate the actual K[1] → error → confrontation and conflict → reorganisation of K
necessity →→→→

4. the question

Here is the question given to the different groups of students:

Let's consider the sets of points of space whose coordinates (x, y, z) are linked by the following relations:

E_1 $2x - y = -1$

E_2 $x = 3$

E_3 $x + y + z = 1$

E_4 $x^2 + y^2 = 1$

E_5 $xy = 1$

Describe and represent these sets of points as precisely as possible.

You can use any method you wish to solve this problem.
Please write the different steps of your research.
Give the result which seems really correct to you and explain how you know that it is correct.
Find the way in which you could convince someone that you result is the correct one.

---

[1] K. stands for knowledge

III. Analysis

1. the work in small groups

The teacher observed the students while they were working alone. At first they all wrote that E_1 was the equation of a line. Some students changed their mind later; it seems that the study of set E_2 was the reason for this change. Of course some other students did not change their mind, so the confrontation was possible[2]. Here are some exemples of what happened in the small groups, as it has been observed by the teacher:

Lorinne started to prove with gestures. She came to the corner of the room and described the position of the three axes: Ox was at the bottom, on the right hand-side, Oy was vertical and Oz at the bottom on the left. She showed, using her right arm, the position of the line $2x - y = -1$ in the plane xOy and said, moving her arm: « *as we are in a space, all points have three coordinates, so my line can move ahead* ». She went on, doing the same gestures and answering the questions of the members of her small group, as long as necessary for them to be convinced. It took quite a long time, but eventually they were convinced.

Laurent used an analogy: « *when we are on a line, x=3 is a point, when we are in a plane, x=3 is a line, here we are in a space, x=3 is a plane.* (The hyperplan is behind this finding, one linear equation for a subspace which dimention is n-1). He went on: « *in x=3, there is no y, in a plan it is still a line and not a point and y can have whatever value we want. Necesserally in a space, even though there is neither y nor z it is a plane. So in a space,* $2x - y = -1$, *where there is no z, z can be whatever we want, it is also a plane* ». This explanation, along with drawings is oriented to his group mates.

Edouard explained why he was convinced with gestures and speach, telling he could see the points: « *piled up one on the top of the others, it can go up, because z has no value* ». Nobody in the group asked him what it meant that *z had no value*.

2.the evolution in the small groups

When the small groups reported about their work in front of the large group, the following steps appeared quite clearly:

- $2x - y = -1$ is the equation of a straight line
- $x = 3$ is the equation of a plane
- come back to $2x - y = -1$, it is the equation of a plane
- $x + y + z = 1$, *I don't know what it is*, or *it is the equation of a plane, by I don't know how to represent it.*

The first sets being studied and correctly identified, the students could turn to the other sets. They started again with the gestures, this time with no difficulty. For $x^2 + y^2 = 1$, they said that it was a tube, a pipe, that you could pile up circles, they could draw it and some found the word *cylinder*. For $xy = 1$, they could not say much about it, but they used gestures and sheets of paper put upright on an hyperbola drawn on the table. It was clear for all of them that it was not a line but a surface.

3. the meaning of the missing variable

We can trace

- *z is not in the equation, so z equals zero,*

---

[2] In one of the large groups no student at all changed is mind. They all believed that the two first sets where lines. Nevertheless when they gathered in small groups and started to explain « how they knew that it was a line » the changes occured.

- *z is not in the equation, so z has no value,*
- *z is not in the equation, so z has any value,* or, *z is not in the equation, so z has whatever value we want, it varies from $-\infty$ to $+\infty$,*
- *there is no z, so z is free,*
- *I would not make mistakes if the equation was written* $2x - y + 0z = -1$.

  This sentence showed clearly that the students had understood that the missing z was not equal to zero, but that it was its coefficient which was equal to zero. And in fact all students agreed to this explanation. In the questionnaire, we found some statements identical to these.

4. four steps in the resolution

This teaching device was ment to bring some changes in the knowledge of the students, through four steps corresponding to the different phases of the work. We shall examine these steps and describe the story of the knowledge:

1. first step: « E_1 is a straight line ». This knowledge comes immediately to the mind of the student, almost in an instinctive way.
2. second step, at the end of the personal work: « E_1 is a straight line » or « E_1 is a plane ». This knowledge can be different for different students; from this difference comes the possibility of a conflict. This knowledge is already the result of some work of the student; it generally comes from the work on the set E_2. As we saw, the students who changed their knowledge can talk about this change.
3. third step, during the work in small groups: « E_1 is a plane ». The knowledge is no longer the knowledge of one isolated person. It is shared by the members of the small group, at least in some groups if not in all of them.The knowledge has an history inside the group as it can be seen by the person who observes the work of the group. The shared knowledge has been constructed through verbalisation or through gestures. This is the time when the students experience the necessity of the mathematical statement: **in an Euclidian space, the equation $2x - y = -1$ has to be the equation of a plane.**
4. fourth step, when the teacher operates in the large group: « E_1 is a plane and it cannot be anything else». At this moment the knowledge is a shared knowledge, even for those who had not constructed it in their small group. There is something more: the teacher, taking argument of what had been said by the students in their reports, makes visible for every one the fact that some groups experienced the necessity of this statement.

5. the role of the fourth step

This fourth step has a most important role in this device, and we will examine it now. We would like to emphasize the fact that this step is made possible by the three previous ones and that without them all it would be impossible for us to reach our aim.

We don't content ourselves with some statement like: « we came to the result that, in a space, $ax + by = c$ is the equation of a plane, as well as $ax + by + cz = d$ », possibly adding a demonstration of it. We also tell the story of this knowledge, in the way that the students experienced it.

If we don't talk about this story, we take the risk that the shared knowledge of the fourth step will very quickly vanish as it has already vanished several times. Remember that our students have learned about equations of planes several times already. We hope that the link between the knowledge and the story of the way in which it had been built, and especially the experience of its necessity, will give this knowledge a special quality and make it more stable.

Conclusion

- It sems impossible to organize such a teaching for each concept that the students have to learn. In fact this would not be useful. We want the students to know that mathematical statements are necessary and that this character of necessity is a quality of mathematical knowledge which cannot be taken apart, otherwise mathematics would not be mathematics anymore. It seems quite obvious to us that, if we take the opportunity to make the students experience this necessity for some statements, they will know that this characteristic is common to all mathematics.Then we can choose some knowledge for which a conflict of the type that we described here can occur. In fact, depending on the level of the students, we work in the same way on different exercices. For instance, with 15 years old, we used the same device to have the students work on inequalities. At this moment, we are working on functions and derivatives with students in the last year of high-school.
- One month after the end of this teaching we gave a questionnaire to the students asking what they remembered of their work, and what had been the most important moments for them during this teaching. Their answers to the questionnaire gave us three types of information:
  First of all, concerning the knowledge itself, they remember that they changed their mind about the missing variable: « *when a variable is not present in the equation, this doesn't mean that it is zero, but, on the contrary, it means that this variable can have any value* ».
  Concerning the necessity, some of them wrote: « *stepping back, and discussing with others, it became obvious to me that* $2x - y = -1$ *had to be a plane* » .
  Finally, concerning the role of *others*, we found this type of remark: « *I found out that, working together, we managed to find the correct answer, although at the beginning we were all wrong* ». This is important to us, as it illustrates the fact that the "mathematical reality" is met through the discussion with others.
- In the introduction, we explained, that the character of necessity of a mathemetical statement has to be experienced, and cannot be just "told" by the teacher. This doesn't mean that the teacher has no role to play in such a teaching device. Quite the contrary. During phase four the teacher does two things: he says what is the correct knowledge and he says that this knowledge could not be different. The first thing could, in some situations, be said by some students who could give a demonstration. When saying this second thing about necessity, the teacher can take argument of the experience that some groups have had of this fact. This makes a difference from a situation where the teacher would say it and no student really knowing what it ment. In this device quite a lot of small groups already know that such a set "has to be a plane". When the teacher says it, he can

refer to this experience which is common to many students. And from that moment on, when the mistake appears in the classroom, it is generally a student who says: « *remember , we all agreed that it could not be a line, it had to be a plane* ». The teacher is no longer the only person who "knows"; the knowledge, together with its necessity, is a common knowledge. In that way, one can say it is a mathematical knowledge.

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INDEXICALITY AND REFLEXIVITY IN THE DOCUMENTARY OF CLASSROOM CONSTRUCTION OF RATIO CONCEPT

Tetsuro Sasaki

Aichi University of Education, Japan

Abstract. The ratio concepts should be analyzed as classroom culture from a Vygotskian perspective or situated learning theory, to which ethnomethodology is indispensable. In mathematical education it is concerned with gaining an understanding of the manner in which school knowledge or classroom culture is accomplished. In the research of mathematical education, indexicaltity and reflexivity from ethnomethodology are identified as important concepts. I describe the documentary, which is also characteristic of ethnomethodology, concerning classroom discussion written by the teacher, who builds an ethnographic context to give his teaching thought a specific sense. And I analyze the classroom construction of ratio concept with indexicality and reflexivity.

1. Introduction

J. Piaget (1966) insists that ratio concept is the critical factor in formal operation and logical thinking. Noelting (1980a, b) shows the developmental stages of strength of juice as a ratio concept. But recently the teacher's or student teacher's ratio conceptions (Klemer, & Peled, 1998; Keret, 1999) and from the social constructivism the class discussion (Pesci, 1998) are studied. Thus for the ratio concept also, analysis of classroom culture from a Vygotskian perspective (Bartolini Bussi, 1995) and situated learning theory (Lave & Wenger, 1991; Alder, 1996) can't be indispensable, which relate to ethnomethodology.

Ethnomethodology is a branch of sociology developed of late years by Garfinkel (1967), which "is concerned with gaining an understanding of the manner in which everyday life is accomplished. Through the disruption of social order, at the level of the everyday and the commonplace, the ethnomethodologist hopes to gain access to the manner in which the social reality of participants is achieved and maintained. (Brown & Dowling, 1998, p.48)" In the research of mathematical education concerning social norms or plans to solve problems, indexicaltity and reflexivity from ethnomethodology are identified as important concepts. Indexicaltity means that the meaning of all behavior and utterance are situated in local historicity. Reflexivity is the mutuality between account and situation.

Then I describe the documentary, which is also characteristic of ethnomethodology, concerning classroom discussion written by the teacher, who "builds an ethnographic context to give his remarks a specific sense (Leiter, 1980, pp. 110,111)." And I consider it with indexicality and reflexivity. In the classroom the extended pseudo idea yields the ratio concept though the discussion.

2. Indexicality and Reflexivity in the Research of Mathematical Education

Yackel & Cobb (1996) identified the importance of reflexivity from ethomethodology, said "the construct of reflexivity from ethomethodology is especially useful for clarifying how sociomathematical norms aad goals and beliefs about mathematical activity and learning evolve together as a dynamic system (p.460)." This dues to the peculiar theory of meaning, according to which "meaning is not a product of a set of internalized rules but is constructed by assembling an ethnographic context of interpretation for occasional expressions" (Leiter, 1980, p.153).

Those are central concepts of the theory of meaning in ethnomethodology. Therefore they should be explained in the first place. Leiter (1980) described indexicality.

> Indexicality refers to the contextual nature of objects and events. That is to say, without a supplied context, objects and events have equivocal or multiple meanings. The indexical property of talk is the fact that people routinely do not state the intended meaning of the expressions they use. The expressions are vague and equivocal, lending themselves to several meanings. (p.107)

And he define reflexivity relating the indexicality as following.

> Reflexivity is ··· a property of social phenomena which, like indexicality, makes social facts the product of interpretation. ··· When defining reflexivity, it is best to remember what makes indexicality an essential property: The contextual particulars are themselves indexical. This sets up the property of reflexivity. Accounts, whether verbal or behavioral, accomplish one thing. They reveal features of the setting to the observer. However, accounts are also made up of indexical expressions, the sense of which depends on supplying ethnographic knowledge of the setting. Accounts and settings, then, mutually elaborate each other. The account makes observable features of the setting - which, in turn, depend on the setting tot their specific sense. The features of a setting that are revealed by descriptive accounts and behavior do not just explicate the setting; they, in turn, are explicated by the setting. (pp.138,139)

In mathematical education the situation and the setting to learn have been recognized as important factor. And recently the significance of situated cognition is gradually approving (Alder, 1996; Cobb & Bowers, 1999). Indexicality and reflexivity mean that describing a setting is constructing it, and expressing it is understanding it, describing doing mathematics is doing itself (Coulon, 1995). This implies that the document of the classroom discourse or discussion is not only described but constructed in research of mathematical education. That is, two reflexivities can be identified at two levels, which is in the classroom and in writing down the classroom event. Sociologist Pollner (1991) distinguish them and call endogenous and radical or referential respectively.

3. Mathematical Documentary of Classroom Construction of Ratio Concepts

The documentary of classroom was carried out and written down by Masataka Kyokushi (1997) who is elementary teacher attached to Faculty of Education Toyama University in Japan. All pupils in this class stayed at the cottage three month ago. Kyokushi (1997) made the pupils remember the experience and posed the problem. In Japan we express the area of room with the number of Japanese mats made of rushes plants - "tatami" -.

《Problem》

Children go on a excursion and stay the accommodation.
The table shows allocation of their rooms. Which room is most crowded?

	the number of mats (area of the room)	the number of children
Room A	8 mats	6 children
Room B	10 mats	6 children
Room C	10 mats	8 children

Moreover he provided the activity of feeling the area of room in the 'work space' so that pupils could imagine it really.

Yukiko answered as following based on this experience.

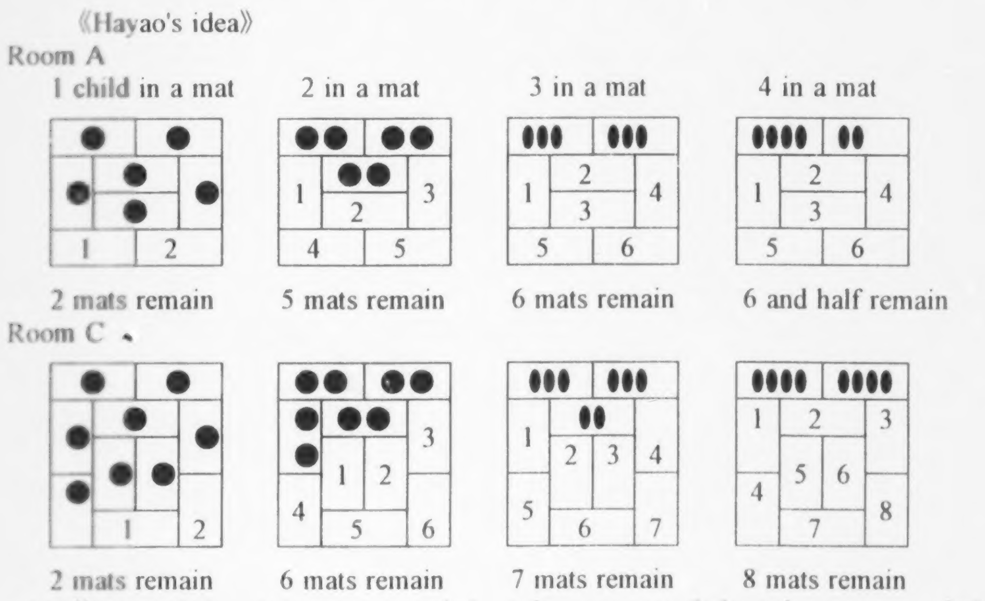
If each children sleeps on a mat (tatami), two mats remain in Room A and Room C, but only in Room B 4 mats.

Room A ··· 8 - 6 = 2   Room B ··· 10 - 6 = 4
Room C ··· 10 - 8 = 2

Room B is least crowded because the most number of mats remain.

Yukiko identified the deference of the number of mats and of children as the criterion of crowd.

For this idea, a pupil said "In this case the difference show that Room B is least crowded clearly but necessarily it isn't the criterion of crowd." This opinion created a stir and led to the arguments both for and against. So The teacher arranged worksheets drawing the room and magnets indicating children to put pupils investigate themselves. In this process Hayao at first thought that Room A and C were equally crowed, where each child lay down on a mat. But he wasn't satisfied with it , for didn't see so in appearance. Then he went to the workspace and looked at the actual room to make sure that two children could lie down on a mat. And he went back to the classroom to consider the case of two children lying down on a mat. In room A five mats remained, in room C six mats. Further he considered the case of three and four ones on a mat and concluded that room A was more crowded than C. Nevertheless he was troubled with different result that they were equally crowed when a child but not when two or three, four on a mat. (Figure 1.)


《Hayao's idea》
Room A
1 child in a mat
2 in a mat
3 in a mat
4 in a mat
1
2
3
4
5
6
2 mats remain
5 mats remain
6 mats remain
6 and half remain
Room C
7
8
2 mats remain
6 mats remain
7 mats remain
8 mats remain
equally crowded
A is more crowded
A is more crowded
A is more crowded

"Hayao was troubled with these difference." *Figure 1.*

Hayao suggested the new idea that more than one children lay down on a mat and judged the criterion of crowd with number not in appearance. Therefore his idea shook not only the thought that both room were equally crowded as the same of mats remained, but that room C was more crowded with the area per unit quantity. When the teacher asked pupils about Hayao's idea, the approvals were 15 and the objections 5, the confusions 20. This number shows that many pupils took the difference as the degree of crowdedness.

After short silence Ryoko and Yuri said as following. Ryoko thought that at first each child held a mat as Yukiko and Hayao, and put their luggage on remained two mats. If a child put it on 1/4 mats, room C was filled, but in room A 1/2 mats remained. Thus room C was more crowded. (Figure 2.)

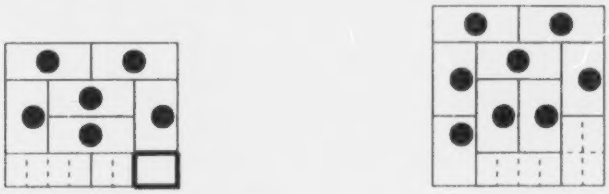
Yuri also put each one on a mat and divided the remained two mats by children.

A ··· 2 ÷ 6 = 0.33···, 1.33··· mats per one.
C ··· 2 ÷ 8 = 0.25 , 1.25 mats per one.
Room C is more crowed.

Moreover Youji said "we should investigate how many mats each child can hold."

A ··· 8 ÷ 6 = 1.33···, 1.33··· mats per one.
C ···10 ÷ 8 = 1.25 , 1.25 mats per one.
Room C is more crowed.
Namely they divided the remained mats equally.

《Ryoko's idea》

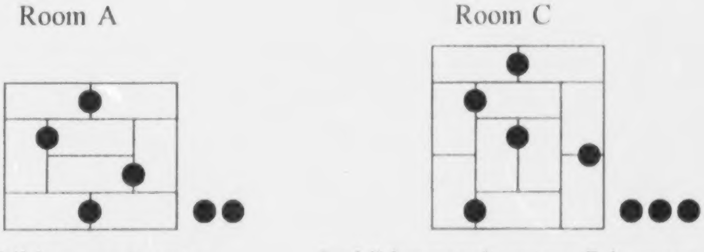


Room A 1/2 mat remains Room C is filled

Figure 2.

Then their ideas troubled Hayao more and more. He admitted the distinction between his idea and their ones which were right, but can't see that his thought was wrong. After this time he wrote "In this discussion I got lost more and more. I can understand Ryouko's and Yuri's thinking, but can't why my thinking is false. So I can't consent" In next hour the teacher told his intention and gave the question whether his thinking is false or not, for all classmates. Then Yuichi said that he contrary to Hayao's thinking of children more than one in a mat, took as the degree of crowdedness how many children couldn't enter the room when mats more than one to each child. As a result room C was more crowded, for the more children not entering the more crowed room. (Figure 3.)

《Yuichi's idea》



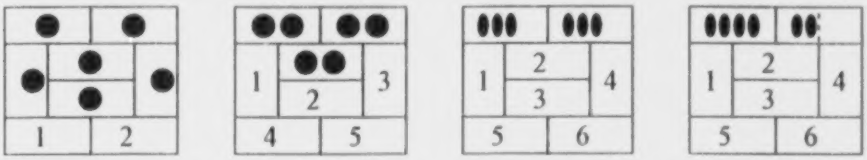
2 children can't enter 3 children can't enter, C is more crowded

Figure 3.

When Hayao caught Yuichi' idea, he eventually understand the remainder of mats or child were not sufficient to compare the crowdedness, it is necessary for all person to divide all of room equally. But realizing his error he was disappointed a little. At that time Jiro advised Hayao, "The idea of increasing 2, 3 ···persons in a mat is't wrong because as Yuri's idea dividing the remained mats with the number of children, we can get the area of mats per a child. (Figure 4.) Before, I thought room A and C were crowded of same degree where same number of mats remained. But Hayao wasn't satisfied with it, and has inquired continuouly. He's great." The other pupils agree with him. Hayao was satisfied with his effort paid off.

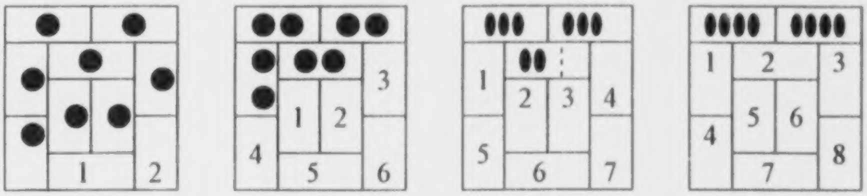
《Jiro's idea》

Room A



	1 child in each mat	2 in each mat	3 in each mat	4 in each mat
	2 mats remain	5 mats remain	6 mats remain	6 and half remain
Given mats at first	1 mat	0.5 mat	0.33 mat	0.25 mat
Divided mats later	2 ÷ 6==0.33	5 ÷ 6=0.83	6 ÷ 6=1	6.5 ÷ 6=1.08
mats per a child	1.33 mats	1.33 mats	1.33 mats	1.33 mats

Room B



	2 mats remain	6 mats remain	7 mats remain	8 mats remain
Given mats at first	1 mat	0.5 mat	0.33 mat	0.25 mat
Divided mats later	2 ÷ 8==0.25	6 ÷ 8=0.75	7.33 ÷ 8=1.92	8 ÷ 8=1
mats per a child	1.25 mats	1.25 mats	1.25 mats	1.25 mats

Figure 4.

4. Consideration of the Mathematical Documentary from Indexicality and Reflexivity

The teacher documented the process of classroom discussion constructing socially ratio concept, indexically, so that he explained the context. Owing to his document we can see the significance of this classroom construction and implication to learn the ratio concept. Leiter (1980) states effect of indexicality.

> Indexicality is not used to point to the utter meaninglessness of the social world,···. Indexicality points to the accomplished nature of meaning. (p.110)
>
> Indexicality and reflexivity (···) are properties of behavior, settings, and talk which make the ongoing construction of social reality essential. Be-

cause behavior is potentially equivocal, people are continually creating its specific sense by embedding it in a context. (p.156)

He posed the problem comparing crowdedness of room, the area per each person. The pupils in this class experienced staying the similar room in a cottage before this time. The experience makes easy for pupils to understand the setting of the problem. To understand problems means to understand its setting. He also applied the indexicality and reflexivity in setting the problem for pupils.

At first, Yukiko suggested comparing their crowdedness with the difference between the numbers of mats and children, which is popular but frequently pseudo way for children. They had to understand why it wasn't sufficient and why the ratio was necessary. It was evident that room A and C were more crowded than B. The question was which was more crowed, A or C. In all process of classroom construction of the ratio concept, Hayao's idea played the most important roll. He extended Yukiko's idea and placed more than one children to each mat, to compare the number of remained mats for crowdedness. But the result that room A and C were equally crowded placing a child on each mat, but room A was more crowded than C placing plural children on each mat, made trouble not only to him but also class. The teacher succeeded in giving to Hayao's statement social reality in the setting, who applied the reflexivity effectively.

Hayao's idea that many mats remained as not held brought up Ryoko's and Yuri's thought of dividing equally the remained mats to all children in the room. Nevertheless Hayao was not convinced of it, because he can't understand why his was false. Then Yuichi suggested the inverse case of placing each child more than one mats and refuted the extended hypothesis of Hayao's idea. Hayao understand it and was satisfied with Jiro' explanation to relate Hayao's idea and ratio and to evaluate his role in classroom discussion.

The teacher remarked that Hayao' idea played the crucial roll to construct socially the ratio concept dividing all mats equally, who "builds an ethnographic context to give his remarks a specific sense (Leiter, 1980, pp. 110,111)" mentioned before. In social construction the error or false thinking plays important roll because it yields essential feature of mathematical concepts. Teachers should try to establish the context from false thinking to mathematical concept with reflexivity among teacher's and pupil's behavior and utterances , thinkings.

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THE EFFECT OF MAPPING ANALOGICAL SUBTRACTION PROCEDURES ON CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Bracha Segalis and Irit Peled
University of Haifa

This study investigated children's spontaneous construction of connections between subtraction procedures in three number domains, and the effect of the procedure structure scheme on their knowledge. The findings show that 40% of the 58 sixth grade children participating in the study were able to identify the mapping between procedures either spontaneously or with very little help. Those who did not make connections on their own underwent a series of mapping instruction sessions. About 75% of the mapping instruction group were able to see the analogical structure of the procedures and improved their understanding and performance in number problems and related word problems.

The effort in mathematics education to promote children's focus on structure and understanding goes together with encouraging children to make connections between knowledge parts. Hiebert and Carpenter (1992) actually *define* understanding as the making of connections and using analogical thinking.

Although it is acceptable that the identification of analogical structures is important (English & Halford, 1995; English, 1997) it is unclear whether children are able to recognize similar structures. In fact, the vast research literature on analogical thinking presents quite pessimistic outcomes. Classical studies on analogical problem transfer (Gick and Holyoak, 1983; Gentner, 1989; Novick, 1988) show that it is difficult to transfer knowledge from an initial (source) problem to another (target) problem. These studies show that not only the identification of analogical structures is difficult but also the detailed mapping between structures is not a trivial matter.

As to analogical thinking in mathematics education, Goswami (1992) claims that it has been unsuccessful. She points out that teachers build upon the assumption that children will spontaneously make the link between the concrete representation and the target concept, an assumption that contradicts the findings in most of the research on analogical reasoning.

In spite of these findings the effort to encourage the emphasis on analogical thinking, i.e. the making of connections, in class instruction continues, with researchers trying to figure out the conditions that might facilitate transfer. English (1997) raises some issues that, in her opinion, warrant attention. She asks whether children have the necessary mental structural components to enable them to reason analogically, whether children know they should look for common structures, and whether they are able to make necessary adjustments in order to apply a solution procedure to a new problem.

This research tries to answer some of these questions. It deals with the structure of subtraction-with-borrowing procedure in whole numbers, decimals, and fractions, and investigates several questions: Can children spontaneously identify structure similarities between these procedures? Will mapping instruction facilitate recognition of structure similarity and creation of a general procedure scheme? What is the effect of procedure similarity recognition on conceptual and procedural knowledge that relates to these procedures? Does the general procedure scheme enable transfer to related word problems.

Our hypothesis were that some children will be able to identify the connections on their own, and others will be able to recognize them following analogical mapping instruction. We also hypothesized that such instruction would also improve conceptual and procedural knowledge within each number domain. Our final hypothesis was that the general procedure scheme will facilitate transfer to related word problems for those who have made the connections on their own, and improve the ability to do so for those who succeeded in the mapping instruction.

The subtraction-with-borrowing procedure was chosen because it involves similar steps and principles in all three number domains.

Similar procedural steps:

1. One has to take away a quantity that is a compound of different units (e.g. ones and tens, "wholes" and fractions, tenths and hundredths) from another quantity of this type. The units are taken away in a certain order (from smallest to largest) by taking the units of the subtrahend from the corresponding units in the minuend.
2. A commonly encountered situation involves not having enough of a certain unit to take away.
3. The problem can be solved by reorganizing the number. Specifically, by changing its representation so that one of the next larger units is exchanged for the missing units.

Similar principles:

1. Number representation can be changed by certain exchange rules: e.g. 10 hundredths for 1 tenth, or 3 thirds for 1 whole.
2. If these exchange rules are kept, then the total amount remains unchanged.

PROCEDURE

A series of interviews was individually conducted with 58 sixth graders, 33 boys and 25 girls, in a middle class urban school. These children had studied the conventional subtraction (with borrowing) algorithm in whole numbers, decimal numbers, and rational numbers (in rational numbers the procedure is not necessarily called "borrowing" although one actually makes "borrowing-type" exchanges).

The study consisted of 4 parts. In the pre-test each student was tested on his conceptual and procedural knowledge of the subtraction procedure in the three number domains and on solving related word problems. Then each student was asked about the similarities between the procedures. Each of these tests consisted of a set of predetermined questions so that the student's performance pre and post instruction could be compared.

On the basis of their performance on a pre-test, students were assigned to different interview routes. Students who spontaneously identified the general similarities of the procedures, or did so after a minimal hint, were assigned to group A, which, later on, did not get any instruction. Students who were not able to see the procedure mapping were divided into two groups, group B who received procedure mapping instruction, and group C who received intra domain instruction.

In the third part of the study all students were interviewed again on conceptual and procedural knowledge of the subtraction procedure, on the mapping issue, and on related problem solving. The last, fourth, part included a retention test on related word problems, which involved unit exchanges (e.g. between meters and centimeters).

Analogical mapping instruction

The procedure mapping instruction started with establishing acceptable subtraction performance, choosing as a base domain the domain in which the child exhibited better competence. It continued with a request to perform the procedure in another domain, the chosen target domain, by using the steps of the base procedure to generate the steps of the target procedure. The third domain was similarly treated, so that eventually all three procedures were performed accompanied by a discussion on the similarities between them.

RESULTS

Our first hypothesis suggested that a reasonable proportion of the students would spontaneously identify connections between procedures. As it turned out, 23 (40%) of the students were assigned to group A including 11 (19%) of them who identified the similarities on their own, and 12 (21%) who made connections following a minor cue.

When asked whether he saw any similarity or relationship between the number problems he had solved, Uri, a child assigned to group A, explained: *They are all subtraction, in all the problems you have got somehow to exchange something for something else.*

The second hypothesis stated that analogical mapping instruction would make an additional part of the children recognize the procedure similarities. This hypothesis was tested on the 26 children in group B, who received analogical mapping instruction of the three procedures. The comparison of children's scores on their similarity test pre and post instruction showed a significance difference. The similarity score was in the range 0-3, the mean score being 0.58 on the pretest, and 2.22 on the post. The change in perceiving the procedure connection is demonstrated in the examples presented in Table 1.

Table 1: Examples of answers on procedure similarity in the pretest and posttest in the mapping instruction group (B).

	Pretest	posttest
Yuval	*No. I see no similarity between them [the procedures].*	*Yes. In each of them I had to borrow from the whole or from the bigger part.*
Sima	*They are almost all fractions. Ah... They are all subtraction, and that's it.*	*They are all subtraction, and in all of them you have to borrow.*

The third hypothesis claimed that those who improved on their similarity perception following mapping instruction, and constructed some general borrowing scheme, would also improve their conceptual domain knowledge. This hypothesis was tested in group B who received mapping instruction and group C, who had intra domain instruction with no mapping between domains. The comparison of conceptual knowledge related to the subtraction procedures in the three domains, whole numbers, fractions, and decimals, resulted in several findings that support the hypothesis: a. Children who acquired a general similarity scheme following mapping instruction also improved significantly in each of the three domains. b. Children who did not acquire a general similarity scheme did not improve in any of the three domains.

However, these results are not strong enough, as the differences between the two B subgroups might come from differences between children, and the improvement might be a training effect rather than the effect of mapping instruction. This methodological issue is answered by testing the control group, group C, on conceptual knowledge that relates to the procedure in each domain. The children in this group did not identify procedure similarity in the pretest. Instead of mapping instruction, they had more instruction within each of the domains. The statistical comparison between their pretest and posttest shows no significant difference in their conceptual procedure knowledge.

The fourth hypothesis suggested that the general procedure scheme facilitates transfer. This means that children who identify connections between procedures would succeed in solving related word problems, or would improve their problem solving as a result of mapping instruction. It also means that the control group, C, who did not have a general scheme and did not get any mapping instruction would perform poorer than group A and worse than the "scheme improvers" in group B (those in B who identified connections following instruction).

Children were given five related word problems, i.e. problems that apply the subtraction procedure. An example of a related word problem: *From an 11 meter long wooden plank a piece of 2 meters and 39 cm was cut off. What was the length of the remaining wooden plank?* Other word problems had measurement units unfamiliar to the children (e.g. yards and feet that are not used in Israel), or more compound units (e.g. weeks, days, hours). The context involved length measurements, time, and value exchange. Each child was scored on word problem performance and on word problem understanding.

The comparison of word problem pretest performance between groups showed, as expected, the advantage of group A. The comparison of pretest and posttest performance within groups resulted in significant difference only for the "scheme improvers" (a subgroup of B). These children improved significantly in word problem performance in length, time and exchange problems. Their word problem understanding improved significantly in time and exchange problems. No significant difference between pretest and posttest was found in any of the other groups. Table 2 presents success rates (percentages) in two of the problems.

Table 2: Examples of change in problem solving transfer (success rate in percentage).

	Problem 1		problem 2	
	Pretest	posttest	pretest	posttest
group A	65	78	67	60
improvers (B)	40	75	47	80
control	38	47	46	47

The results in these problems demonstrate the superiority in pretest performance of children who identified the general scheme over the two other groups. The two latter groups performed similarity in the pretest, yet only one of them improved in the posttest, the group of "improvers" who acquired a general procedure scheme.

It is interesting to note the strategies that were used by the children in solving the word problems. All problems could be solved by subtraction using some unit exchange. Most of the children used a strategy that called for understanding the problem situation and using the subtraction scheme. The most often used strategies were :

1. Exchanging all given units to the smallest unit (e.g. all kg. into gr.).
2. Dealing with the units one at a time.
3. Using a vertical alignment of corresponding units and subtracting as in whole numbers.
4. Representing the units as fractions or decimals and using the corresponding subtraction procedure.

The connections made by children between the problem structure and the subtraction procedure involved coping with problematic situations. Thus, for example, some children came across an obstacle in trying to use a fraction representation while solving the following problem:

In a boat sailing contest Avi was the first to arrive after sailing for 1 week, 5 days and 18 hours. Oded came in last after sailing for 2 weeks, 3 days and 4 hours. What is the difference in time between them?

Zohar explains her deliberations: *The trouble is – how can I use a week, 5 days and 18 hours? Here I have 3 given numbers, but in fractions I only have 2 places, that of the wholes and a place for the fraction. Interviewer: So what will you do? Zohar: I exchange the week into days.*

Ido, who solved an earlier problem using a decimal representation, tried to use the same method in this problem, but realized that something was wrong. He solved it again using fractions and explained: *At first I wanted* *to use a decimal fraction, but a decimal fraction does not go with the 7 [days per week] here. So I did not do it. Then I thought of fractions. In fractions I only had one place.... As though for the hours I had no place. So I moved the week and the days and added them [together], because there are 7 days in a week,...[so I got] the number of days, and the 18 hours I did [turned into] a fraction.*

DISCUSSION

This study focuses on the following questions: a. Can children make connections between procedures that have a similar structure? b. Can mapping instruction facilitate the construction of a procedure connections and creation of a general procedure scheme? C. Do children *benefit* from constructing or acquiring this general scheme?

In spite of the pessimistic previous research on this issue, forty percent of the sixth graders participating in this study had some general subtraction scheme or were able to construct it with very little help. In addition to this, most of the rest of the children were able to acquire the general subtraction scheme following mapping instruction. These results are quite surprising and encouraging. In view of the fact that procedure similarities are not discussed in class often enough, it is surprising that children make connections on their own. In view of research showing that mapping between structures is considered a non-trivial matter, the effect of mapping instruction is encouraging.

With the findings that indicate that mapping instruction improved children's ability to identify connections, the contribution of this guided learning of connections had still to be investigated. The results showed that children in the mapping instruction group, who managed to create a general procedure scheme, also improved their conceptual and procedural knowledge of the three domain procedures. These children, who performed worse than group A (those who identified the connections spontaneously) in transfer word problems, improved in their problem solving to the extent that they became quite similar in performance to group A children.

These results imply that although mapping instruction involves a guided construction of a general scheme, it can effect children's knowledge quite similarly to a self constructed scheme. Further research might look into the effect of this instruction on improving children's ability to identify connections on their own.

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A TEACHING EXPERIMENT ON MATHEMATICAL PROOF: ROLES OF METAPHOR AND EXTERNALIZATION

Yasuhiro Sekiguchi
Yamaguchi University, Japan

A teaching experiment was conducted on mathematical proof. The method developed for teaching proofs had three components: (1) "Adventure" metaphor was stressed in introducing mathematical proof, (2) every student was asked to make a "file" from the worksheets, and (3) group activities of presenting proofs were organized. The classroom teacher taught the lessons for about five months. The metaphorical introduction was effective for making learning of proof understandable and enjoyable. The use of "file" played significant roles for classroom processes and student learning. The group activities needed deliberate guidance by the teacher.

Backgrounds of the Study

This paper reports a teaching experiment about the instruction of mathematical proof. The concept of mathematical proof is one of the most difficult topics for junior high school students in Japan. The current study developed a method for teaching proofs to improve students' conceptual and affective difficulties in learning proof. There were three main theoretical backgrounds in this study.

Metaphorical Mapping

Learning is a meaning construction process inside student's mind interacting with the other people and environment. According to Lakoff(1987), the constructions are mediated by cognitive models. Cognitive models are generated by two kinds of preconceptual structures: They consist of two types:

A. Basic-level structure: categories formed from gestalt perception, bodily movement, and mental images.
B. Kinesthetic image-schematic structure: categories formed from relatively simple structures recurring in everyday bodily experiences.

The cognitive models are constructed through projections using metaphors or metonymies from these preconceptual structures.

According to Johnson (1987), and Fauconnier (1994, 1997), mappings are essential in meaning construction; they work for connecting and generating "mental spaces." Fauconnier (1997) identifies three types of mapping: Projection, pragmatic function, and schema mappings. One of the most important projection mappings in human understanding is metaphor (or analogy): It connects a familiar domain ("source") to another domain ("target"), which is the focus of attention, with a guide (schema mapping) of common cognitive model (pp. 102-105).

Use of metaphor in introducing mathematical proof has been little explored in research. Various metaphors could be used in conceptualizing mathematical proof

(Sekiguchi, 1999), however. In this study, "adventure" metaphor was chosen in introducing mathematical proof: The source is adventure, the target is proof, and the common cognitive model is JOURNEY schema (Lakoff, 1987). This choice was because adventure metaphors were often used for conceptualizing mathematical problem solving in general (cf. Chapman, 1997), and students seemed familiar with them, and because the popularity of adventure seemed to help students feel that doing proofs is exciting and enjoyable.

JOURNEY schema is a kinesthetic image-schematic structure from such everyday experience that "[e]very time we move anywhere there is a place we start from, a place we wind up at, a sequence of contiguous locations connecting the starting and ending points, and a direction" (Lakoff, 1987, p. 275). This consists of four elements: "a SOURCE (starting point), a DESTINATION (end point), a PATH (a sequence of contiguous locations connecting the source and the destination), and a DIRECTION (toward the destination)" (p. 275). In the current study I extended the metaphor by introducing two new components to the source and target domains (Table 1): TOOLS (available means) and IDEAS (journey plans). I expected that this extension facilitates students' analysis of their own reasoning processes.

Table 1
Correspondence Between ADVENTURE and PROOF

	ADVENTURE	PROOF (EXPLANATION)
SOURCE	a starting location	a given problem situation
DESTINATION	a goal location	a new theorem
PATH	a course of moving from the start to the goal, filled with a lot of danger and excitement	a process of reasoning
DIRECTION	toward the goal	toward a new theorem
[TOOLS]	physical vehicles, instruments, or devices available	conditions of the given problem, definitions, postulates, already proved theorems
[IDEAS]	plans, strategies, or tactics to cope with difficult situations	plans, insights, intuition, or strategies to deduce the theorem

For a metaphor to be effective, one needs to be familiar with its source domain (see English, 1997). In a lesson of this experiment we instantiated the source domain by referring to adventures in a cartoon series "Doraemon"(Magic Cat) which was very popular among the students, and discussed correspondence between adventure and explanation. After that, we introduced the term "proof" for mathematical explanation.

Externalization

One of the important roles of mathematical proof is systematization of mathematical propositions: Proofs connect definitions, postulates, and theorems each other by illuminating logical relationships among them. This systematization keeps mathematical investigation from becoming just a collection of trial-and-error efforts.

This study did not use any commercial textbooks in class. Instead, we prepared numbered worksheets for lesson. Every student was asked to glue the worksheets together in the order of number so that making an ordered single "file." It was designed to help students systematize mathematical propositions discussed in lesson.

A worksheet in most cases contained a few problems, figures, or statements. We always put large blank spaces in a worksheet for students to be able to write their own thinking, ideas, solutions, and advice, as well as the teacher's and other students' suggestions. I expected that this writing in the files would help students to "externalize" their own learning, that is, to create those materialized objects which enable them to communicate their learning with other people. Bruner (1996) summarizes important potentials of "externalization":

> Externalizing, in a word, rescues cognitive activity from implicitness, making it more public, negotiable, and "solidary." At the same time, it makes it more accessible to subsequent reflection and metacognition. (pp. 24-25)

In Fawcett (1938)'s famous experiment also, students created their own textbooks "A Theory of Space" during the course:

> The teacher discouraged any attempt by the pupils to memorize the definitions and assumptions accepted. On the other hand, each pupil was encouraged to use his text freely and to refer to whatever definitions and assumptions he needed in the development of his work. This served to emphasize the importance of his text and was a strong factor encouraging him to keep it neat, well organized and always up to date. As new definitions and assumptions were made they were written in the text with numerous illustrations and supplementary comments, depending on the interests and abilities of the individual pupils. (p. 45)

I believe that this practice was crucial to the success of Fawcett's experiment, though it has not been fully appreciated by researchers. Without using the notebooks, students would not have been able to keep track of what they discussed and accepted in the class. Also, writing down their thinking would have helped them to reflect on it, so that enhancing their critical thinking, the main goal of Fawcett. Unlike commercial textbooks, the notebooks were created by students themselves. This must have facilitated students' awareness of their responsibility for their learning.

Social Interaction

As another type of externalization, on two occasions, activities for group presentation were held in class. Students worked on proof problems in small groups, and each group was given a role of presenting solutions to the class or asking questions to them. They were special events for students to work cooperatively in a group, explain their ideas to the other people, and argue with them. I expected that presenting and discussing in class would facilitate students' reasoning and writing. Vygotsky's developmental theory states that intermental activities facilitate the growth of intramental abilities. I believed that presentation and discussion in small groups and in the whole class would stimulate the internal growth of mathematical reasoning, and that oral explanations to the other people would facilitate development of student's ability of written explanation.

Methodology

The teaching experiment was conducted from the end of October 1997 through the beginning of February 1998 at a junior high school in Yamaguchi, Japan. A teacher volunteered to participate in the study. He was teaching five eighth grade mathematics classes, and allowed me to use them as sites for the study. Because observing all five classes were practically impossible, I selected one of them, class A (32 students), as the focus of the data collection, though data of the other classes were also collected when I considered relevant. The criteria of the selection were (1) that the mathematics schedule of class A fitted my schedule, (2) that class A seemed to contain various types of students, which enabled me to collect a wide variety of data, and (3) that the students seemed very cooperative with the teacher, so with the research.

I went to the school almost everyday. The teacher and I discussed lesson plans and developed lesson materials. The lessons were about geometry, and mathematical proof was one of the main topics. He taught the lessons, and I observed and helped students, taking notes, and audiorecording the lessons. After each lesson, we talked about the lesson, the lesson materials, students' responses, and next plans. On the occasions where small group activities and group presentations were organized, some of the lessons were recorded by video camera. The students' files were collected and examined in December and February. A questionnaire about the lessons was given to the students in February. The data were mostly analyzed qualitatively.

Description of the Process of Teaching Experiment

The idea of making a "file" was introduced at the first lesson of the geometry units. The teacher explained to the students that the file had the roles of notebook, textbook, and reference books. He encouraged them not to erase their own solutions but edit them by inserting comments in balloons, or making corrections with colored pens; he asked students not to just copy down the solutions presented at the board. He stressed that it was important to respect one's own ideas and expressions whether or not they were correct, and that there would be more than one way of solving a problem and expressing solutions. He handed out a new worksheet about every two lessons.

The first lesson discussed why the sum of angles of a triangle is 180 degrees. Many students tried to show it by cutting off the angles of a paper triangle, and arranging them on a line. The teacher suggested use of properties of parallel lines, and moved on to the unit of parallel lines. Thereafter, the class discussed how to *explain* why the sum of angles of a triangle is 180 degrees by using properties of parallel lines. In the next several lessons the teacher gave several problems to explain by geometric properties.

In these explanation problems, we adopted a special format of writing (I call it here "two-level format"): We divided the blank space below a problem into two columns by drawing a broken vertical line, and asked the students to write their own explanations in the left column (First level), and use the right column (Second level) for writing the teacher's advice, other students' ideas, their own corrections, figures to illustrate ideas, and so on (This is different from the "two-column form," see Sekiguchi, 1999). I expected that the right column would facilitate student's metacognitive activities. The teacher several times demonstrated how to use this format during lesson, and handed out a copy of a student's writing as a model.

In explanation problems, the teacher several times picked up a student's writing, and discussed whether it was easy to understand, and how to improve it. He showed how to "edit" the student's original writing at chalkboard (Figure 1), and asked students to edit their own writing by themselves. Main goals of editing were to make one's writing understandable by other people, to remove ambiguities, and to emphasize important points. He encouraged them to explain solutions by using words.

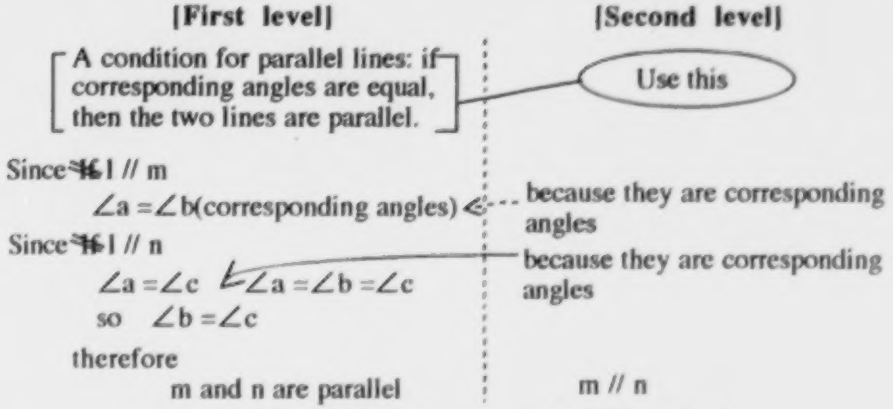

[First level]
[Second level]
A condition for parallel lines: if corresponding angles are equal, then the two lines are parallel.
Use this
Since l // m
∠a = ∠b(corresponding angles)
because they are corresponding angles
Since l // n
∠a = ∠c
∠a = ∠b = ∠c
because they are corresponding angles
so ∠b = ∠c
therefore
m and n are parallel
m // n

Figure 1. An example of two-level format writing.

After students worked on many explanation problems, we introduced the metaphor of adventure to discuss the concept of mathematical explanation. Then, the teacher introduced the conditions for congruent triangles, and discussed explanation problems which required to use those conditions. At that time he told students that mathematical explanation was called "proof." He discussed what were tools and ideas in proofs. From that time on, he often used the adventure metaphor when discussing proofs, and reminded of it the students. After the second term exam, the class discussed several definitions and properties related to isosceles triangle and right triangles. Working on several proof problems, the class held an activity for group presentation. Then the lesson moved on to the unit of properties of parallelogram. At the end of the unit, we held another activity for group presentation. We then conducted a questionnaire survey, ending the classroom observation. At the end-of-year exam we gave several proof problems. After the exam we collected the students' files.

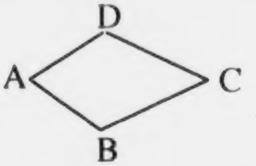
Results and Discussion

The data analysis of the teaching experiment are discussed below with regard to the three theoretical components of the study.

Understanding of the Adventure Metaphor of Proof

To assess the students' understanding of the adventure metaphor of proof, the data from class A were analyzed. At the second term exam, the following problem was given (Figure 2):

Proof (explanation) is "an adventure journey." Answer to the following questions about the proof written below:



When AB = AD and BC = DC,
$\angle ABC = \angle ADC$

[Proof] Draw a segment between A and C.
From the problem, AB = AD
BC = DC.
Also, AC = AC.
Because three sides are equal, respectively,
$\triangle ABC \equiv \triangle ADC$.
Because the corresponding angles of congruent figures are equal,
$\angle ABC = \angle ADC$.

(1) What "tools" are used here?
(2) What "ideas" are used here?
(3) Where is the "goal"?

Figure 2. An assessment problem at the second term exam.

For (1), the majority of the students considered the condition for congruence of triangles (this case, SSS) as a "tool." The next most was the conditions of the problem. For (2), what students considered as "ideas" the most was "connecting A and C." The next most was "the measures of the corresponding angles of congruent figures are equal." The third most was "AC = AC." They are all what were not written in the given problem. There were a few overlaps with the answers of (1): some students selected the conditions for congruent figures. This indicates confusion of tools and ideas. For (3), the students chose the statement to prove as the goal.

At the end of year exam also, a problem asked the students to write ideas and tools of a given proof. The responses indicated that most students were able to identify ideas and tools in a given proof.

The questionnaire asked the students to describe what tools and ideas of proof were like. Most students understood that tools were what one "needs" in proving or what help them in solving (proof) problems: (S7)"Indispensable things when making sure of congruence or gathering [suitable] conditions in doing proof," (S31)"They play important roles that help us when solving problems. I think they will help us in the future, too, so I am going to solidly memorize them." [Note: S[a number] is pseudonym of a student]

Most students understood that ideas are something they have to "think" and find, other than the given tools, in proving: (S7)"Ideas are to draw what one wants to solve using tools by one's own way," (S44)"They are what one discovers or figures out by oneself on a problem or during solving it, so they are needed in solving problems."

Some of the above data and class observations indicated that several students experienced difficulties in distinguishing between "tools" and "ideas." Especially, capturing ideas seemed to be rather difficult. I believe that "ideas" belong to higher cognitive levels than "tools." Students need to work on more proof problems to be able to isolate and describe "ideas."

Affective aspects of using the metaphor were indicated. In the questionnaire, most students expressed that the metaphorical introduction of the concept of proof made learning of proof more understandable and enjoyable: (S5)"Math is often difficult, but when likened this way, I feel as if math becomes easier," (S8)"It's a very good simile. In an adventure you cannot know what happens next until the end. Projecting such excitement into proof, we can do proof more enjoyably. It's a good simile."

Roles of the File

Classroom observations showed that the teachers and students were able to use the files as common resources in lesson, and integrated them into the classroom mathematical discourse. The students personalized the files, on the other hand: they wrote on the worksheets in their own ways, and decorated them by putting marks, using colors, drawing cartoon characters, and so on. It seems that the files worked as a mediating tool between the classroom discourse and individual's thinking.

The use of worksheets actually saved class time of drawing figures on their notebooks, so that the class was able to spend more time on working on proofs. In the questionnaire, many students indicated their appreciation of practical value of the file:

Useful in reviewing: (S7)"Because important points are summarized in the file, it was useful in reviewing and studying very much."

Easy to take notes: (S5) "Not copying down everything on the board, but we just had to write important points. It was easier."

Saving time: (S41)"If we use notebooks, we would have to spend lots of time in drawing figures and writing problems. But, since the file contained them already, it was helpful."

Interaction at the Group Presentation Activities

During preparation for the presentation, the students of each group worked together to find a solution of the assigned proof problem, and discussed with the group members how to present the solution. This preparation process seemed much more interactive than the presentations themselves. In fact, in the questionnaire students showed an appreciation of the group cooperation. The presentation was the goal of the group activity, but became rather ritualized. Also, understandable presentation was not possible without deliberate preparation and the teacher's guidance. And, students expressed difficulty in asking questions to the presenters. As a result, little argumentation occurred between the presenters and audience. Affective and social obstacles appeared to be involved (cf. Lampert, Rittenhouse & Crumbaugh, 1996).

Concluding Remarks

Explicit use of metaphor in introducing mathematical proof is not common. Many metaphors are involved in mathematics teaching (English, 1997), however. We need to be conscious of them, and reconsider what metaphors are appropriate. Though this study emphasized an "adventure" metaphor, any single metaphor cannot handle all the aspects of proof. What metaphor to use, for what purpose to use it, and how to instantiate it in a particular situation need to be closely explored.

Students' writing of files contained rich expression of their thinking. Usual symbolic writing in mathematics may appear simple and elegant, but it is not always the case for students, and has a danger of planting in students' minds an inappropriate belief: Use of words is not important in writing and explaining mathematics. We need to further study about how to facilitate students' writing of proof.

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METACOGNITION: THE ROLE OF THE "INNER TEACHER"(6)
Research on the relation between a transfiguration of student's mathematics knowledge and "Inner Teacher"

Keiichi SHIGEMATSU,NARA University of Education, Japan
Yoshio KATSUMI,Nara Municipal Board of Education, Japan

ABSTRACT

The nature of metacognition and its implications for mathematics education are the main concerns of our investigations. We argued in the last five papers that "metacognition" is given by another self or ego which is a substitute for one's teacher and we referred to it as "inner teacher".

In this paper, we investigate more deeply the concept of the inner teacher focused on the relation to a student's mathematics knowledge through the analysis of his/her learning processes of mathematics in elementary school using the two different methods which are magnetic name cards and student's journal writing.

We found that there are some metacognitive skills and knowledges which influence the student's mathematics knowledge.

In the case study, the student implied the existence of specific steps of the process of forming his/her own mathematics knowledge.

AIM AND THEORETICAL FRAMEWORK OF THE RESEARCH

1. Definition of "metacognition" and "inner teacher"

We are often inclined to emphasize only pure mathematical knowledge in education. And we fail to enact it in students. Consequently, they fail to solve mathematical problems and forget soon after paper and pencil tests.

Recently, "metacognition" has come to be noticed as an important function of human cognitive activities among researchers of mathematics education as well as among professional psychologists. But even so, the definition of "metacognition" is not yet firmly settled, and results from the research have been of little use to the practice of mathematics education.

The ultimate goal of our research is to develop clear conceptions about the nature of "metacognition" and to apply this knowledge to improve methods of teaching mathematics and teacher education. This paper is one of a series of studies in pursuit of this goal.

Roughly speaking, we could regard "metacognition" as the knowledges and skills which make the objective knowledges active in one's thinking activities. There are a few proposals on the categorization of "metacognition" in general, but here we will follow the suggestion of Flavell and adopt four divisions of metacognitive knowledge of:

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(Metaknowledge)

1. the environment   2. the self
3. the task   4. the strategy

and three divisions of metacognitive skill:

(Metaskill)

1. the monitor   2. the evaluation
3. the control

Our unique conception is that this "metacognition" is thought to be originated from and internalized by the teacher him/herself. Teachers cannot teach any knowledge per se directly to students but teach it inevitably through their interaction with students in class.

We start from a very primitive view that teaching is a scene where a teacher teaches a student and a student learns from a teacher. In the process of teaching, a phenomenon which is very remarkable from a psychological point of view will soon happen in the student's mind; we call this the splitting of ego in the student, or we may call it decentralization in a student, using the Piagetian terminology. Children, as Piaget said, are ego-centric by their nature, but perhaps as early as in the lower grades of elementary school, their egocentrism will gradually collapse and split into two egos: the one is an acting ego and the other is an executive ego which monitors the former and is regarded as the metacognition. Our original conception is that this executive ego is really a substitute or a copy of the teacher from whom the student learns. The teacher, if he/she is a good teacher, should ultimately turn over some essential parts of his/her role to the executive ego of the student. In this context, we refer to the executive ego or "metacognition" as "the inner teacher" .

The advantage of this metaphor is that we could have the practical methodology to investigate the nature of metacognition; that is, we may collect many varieties of teachers' behaviors and utterances in lessons and carefully examine and classify them from some psychological view-points.

2. Positive and Negative Metacognition

For Metacognition, we think that there are two types. One is a positive metacognition that promotes positively students' problem-solving activities. The other is a negative one that obstructs their activities. For example, most students believe that statements like questionnaire "When you get lost while solving the problem, please think of other strategies." help them and have a positive effect on problem-solving. This item shows metacognitive knowledge of strategy for problem-solving. This works according to the monitor "I have lost my ideas for the next step". A metaskill of control "Please think of other strategies." works successfully according to a logical conclusion of modus ponens from two premises; the above item and the monitor. (See Hirabayashi & Shigematsu, 1987, for more detail.) Other statements, like "Can't you solve this easy problem?", are believed to make students do worse and to have a negative influence on problem-solving.

3. Hypothesis of students' learning process in relation between a transfiguration of student·s mathematics knowledge and "Inner Teacher"

Several articles have focused on student's mathematical knowledge (solution strategies) (e.g. Carpenter,Fennema & Franke,CGI,1996), but we can't find the articles which analyze the relation between a transfiguration of student·s mathematics knowledge and "Inner Teacher"

For the students' learning process of a student·s mathematics knowledge, we think that there are five stages as follows;

1) Private experienced knowledge

When a student learns a topic of mathematics in the lesson, he/she notices his/her own knowledge referring to everyday·s experiences and a knowledge owned by proceding learning at first.

2) Private heuristic knowledge

After a student has noticed his/her private and rough knowledge, he/she solve the problem of the today·s topic and get his/her own answer as a knowledge.

3) Private discussed knowledge

After almost of students get their private heuristic knowledge, students propose their solutions sometimes orally, sometimes on a small or large chalkboard or a small sheet of paper. A teacher definitely want to compare several students' ideas by themselves. During this discussion, a student get his/her modified knowledge.

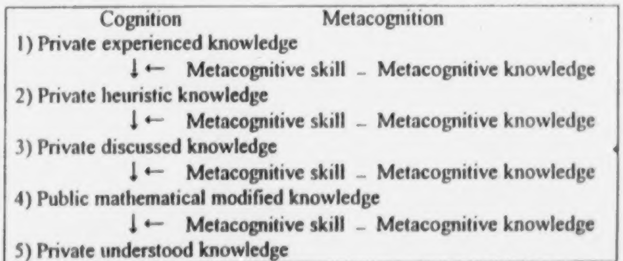
4) Public mathematical modified knowledge

After students discuss with help and guidance by the teacher, the teacher summarizes the day's mathematical idea using or referring to students' private discussed knowledge and comments on other students' private discussed knowledge and what differences there are between them. At this stage, a student can get a public mathematical modified knowledge.

5) Private understood knowledge

After assigning exercises, a student can get his/her own modified private mathematical knowledge. In many cases, students' private understood knowledge will be changed through a learning unit and sometime later.

The transfiguration is roughly imaged as follows;



Cognition Metacognition

1) Private experienced knowledge

↓ ← Metacognitive skill – Metacognitive knowledge

2) Private heuristic knowledge

↓ ← Metacognitive skill – Metacognitive knowledge

3) Private discussed knowledge

↓ ← Metacognitive skill – Metacognitive knowledge

4) Public mathematical modified knowledge

↓ ← Metacognitive skill – Metacognitive knowledge

5) Private understood knowledge

Fig.1 relation between a transfiguration of student·s mathematics knowledge and "Inner Teacher"

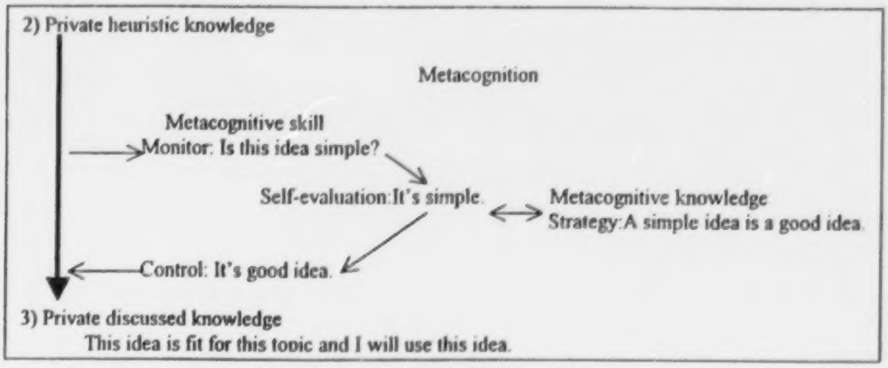

2) Private heuristic knowledge
Metacognition
Metacognitive skill
Monitor: Is this idea simple?
Self-evaluation:It's simple.
Metacognitive knowledge
Strategy:A simple idea is a good idea.
Control: It's good idea.
3) Private discussed knowledge
This idea is fit for this topic and I will use this idea.

Fig.2 Example of a transfiguration 2) to 3)

METHODOLOGY OF THE RESEARCH

1.Teaching-Learning Process of Experimental Lesson in Class using Magnetic Name Card

(1) Teacher presents some familiar topics for students
(2) Reviewing the previous day's problems
(3) Introducing a topic as a style of problem-solving usually
(4) Understanding the topic
(5) Problem solving by students, working in pairs or small groups
(6) Comparing and discussing: problem-solving by class

During discussing, students can move their magnetic name card to more similar idea and record it at their private note if they change it. Teacher record this students' activity using a video tape recorder.

(7) Summing up and generalizing by teacher
(8) Students' record(Journal)

After teacher's summing up, students write their idea of today's topic and why they change it during a class.

(9) Assigning exercises

2.Method of Analysis of the Process - Student's journal writing

At the end of lesson, teacher give about 5 minutes for students to write their writing about today's learning according to teacher's indication as follows;

> 'Please write journal about today's learning especially focused on the process of your brains.'

3.Interview - Case Study

After checking students' journal, teacher picks up some students as Case

Study.

Data collection

We collected the data from students of elementary school as follows;

Table1: Data collection

Case	Grade	Location	Number of Students	Date
1	5	Nara City	36	1996.6.5
2	6	Nara City	36	1997.6.23-30
3	4	Nara City	29	1998.5.21-27
4	4	Ikoma City	35	1998.5.21-27

Table2: A classification of the data

	Term	
Method	A Lesson	A Unit
Magnetic Name Card	1	2,4
Journal		3

RESULTS AND DISCUSSION

1.About Quantity of a student's mathematics knowledge

There are few metacognitive comments at students' note of the step '2) Private heuristic knowledge'. On the other hand, we could identify many metacognitive comments after the step '3) Private discussed knowledge' because students write the reason why they hit on an idea.

Students' metacognitive comments are mainly strategy.

2. About Hypothesis of a student's learning process of mathematics knowledge

The result of Case1

The problem is as follows,

624*32=19968, in the case of 6.24*32=?

1) About Private experienced knowledge

In this case, we couldn't identify this knowledge at students' note because they replied quickly and mainly orally.

2) Private heuristic knowledge

① 6.24 is 1/100 of 624 (6/36)
② 6.24 has two digit in the fractional part (16/36)
③ Strategy of paper and pencil (13/36)
④ Estimation (1/36)

3) Private discussed knowledge

We could identify some students' various knowledge.

① (10/36) ②(15/36) ③(9/36) ④(1/36)

4) Public mathematical modified knowledge

Teacher gave comments on each students' private discussed knowledge about what differences there were between them.

5) Private understood knowledge

In this case, we couldn't also identify this knowledge.

3. Case Study of Relation between a transfiguration of student's mathematics knowledge and "Inner Teacher"

In this case study, we can identify some different cause for a transfiguration.

1) Referring to metacognitive knowledge of strategy and task
Private heuristic knowledge ②
↓ ← Metacognitive skill – Metacognitive knowledge
For example, This idea is simple.
I can solve this problem exactly using this idea.
Private discussed knowledge ④→③
↓
Public mathematical modified knowledge ③
↓ ← Metacognitive skill – Metacognitive knowledge
For example, I can solve this problem well using this idea.
Private understood knowledge ③
2) Referring to metacognitive knowledge of environment
Private heuristic knowledge ③
↓ ← Metacognitive skill – Metacognitive knowledge
For example, This idea is good because of Mr. K's idea..
Private discussed knowledge ③
↓
Public mathematical modified knowledge ③
↓ ← Metacognitive skill – Metacognitive knowledge
For example, K's idea isn't wrong and bad.
Private understood knowledge③

Fig.3: Case Study of Relation between a transfiguration of student's mathematics knowledge and "Inner Teacher"

4. Result of observation for a unit of learning

1) When students' must have a chance that they can solve the problem using the idea, students' knowledge become stable with the metacognitive skill 'do well'.
2) Only when students encounter the problem that they can not solve during a unit of learning, they think they must change their idea with the metacognitive knowledge 'we must think another idea when we can't solve the problem'.

5. Difference between a magnetic name card and journal writing

1) Using a magnetic name card, we, teacher, can catch students' idea and its relation with the metacognition during the process of students' thinking.
2) In the method of a magnetic name card, we can mainly identify students' metacognitive comments concerning the strategy. On the other hand, we can identify students' metacognitive comments concerning the strategy, task and self using journal.

For example, it's great that we devise a idea which nobody devise.

Because students review their metacognitive aspect concerning with the task and self.

6. Teacher's comments orally and with red pencil in students' journal writing

Students' can't write their metacognitive comments by themselves at first. At that time, teacher must support timely and give some appropriate comments as follows;

- please write your any thinking comments freely at your note.
- please write your any thinking comments concerning with well-understood and interesting.
- please write your any thinking comments concerning with your willness to do next.
- please write your any thinking comments concerning with your cause of idea.
- please write your any thinking comments coming with your behind of the brain.

CONCLUSION

In this paper, we investigate more deeply the concept of the inner teacher focused on the relation to a student·s mathematics knowledge through the analysis of he/her learning processes of mathematics in elementary school using the two different methods which are a magnetic name card and student·s journal writing.

At first, we proposed the hypothesis of students' learning process of mathematics knowledge and the relation between a transfiguration of student·s mathematics knowledge and "Inner Teacher". According to this hypothesis, we implemented the experimental lesson and middle term unit of lessons that students solved the process-problem. After the lesson and unit, students got their modified knowledge according to some metacognitive knowledges using magnetic name card and students' journal.

We obtained several findings as follows:

1. We found that there are some metacognitive skills and knowledges which are

connected with the student·s mathematics knowledge.

There are few metacognitive comments at students' note of the step '2) Private heuristic knowledge'. On the other hand, we could identify many metacognitive comments after the step '3) Private discussed knowledge' because students write the reason why they hit on an idea. Students' metacognitive comments are mainly strategy.

2. Only when students encounter the problem that they can not solve during a unit of learning, they think they must change their idea with the metacognitive knowledge 'we must think another idea when we can't solve the problem'.
3. We found some differences of checking metacognition using methods between a magnetic name card and journal.
4. It's very important for a teacher to give some comments orally and with red pencil on students' journal.

But we know that this is not enough to analyze the process of students' learning of mathematics knowledge with metacognition. In order to identify these processes, we need more experimentation on this issue.

Acknowledgement

We wish to express thanks to the students who participated in the study.

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AN ANALYSIS OF "MAKE AN ORGANIZED LIST" STRATEGY IN PROBLEM SOLVING PROCESS [1]

Norihiro Shimizu
Fukuoka University of Education, Japan

In this study, 6th grader children's problem solving processes concern with "Make an organized list" strategy is examined in the context of problem solving. Seven 6th grader in the study are required to solve the "Make-15-by-three-cards" problem that requires children to use "Make an organized list" strategy. From the data, children's errors, factors of successful/ unsuccessful behavior in solving this problem and difficulties of this problem were analyzed. As a result, errors such as "misunderstanding the problem", "erroneous multiplication" and "incomplete elimination of extra components" are detected. Finally, this study suggested some kind of metacognition or alternative elimination method to solve the problem.

1. Introduction

Many researchers have acknowledged that problem solving strategies are playing important roles in successful problem solving (e.g. Schoenfeld,1985). And, recently, the trend of research methods in problem solving strategies had changed from quantitative (statistical) method to qualitative one. For example, Nunokawa (1997) had examined how solvers would use solutions of simpler problems to explore original problems ("Use similar and simpler problems" strategy). In another study, he examined the role of "Draw diagrams" strategy in mathematical problem solving (Nunokawa,1994).

Although many strategies have been identified as ones to be taught, some strategies weren't examined in the context of problem solving but as a specific topic. As an example, "Look for a pattern" is an important problem solving strategy, but recently it seems to be studied in the context of "algebraic thinking" or "generalization of pattern" rather than as problem solving (strategy). "Make an organized list" strategy is also important one, but its usage in problem solving process is not well examined. Rather, it is associated with solving combinatorial tasks (for example, usage of tree diagrams).

2. Research Questions and Method

Research Questions

As mentioned above, the process and the usage of "Make an organized list" strategy isn't fully explored. In this study, 6th grader children's problem solving processes concern with "Make an organized list" strategy is examined in the context of problem solving.

The purpose of this research is to discuss the following:

- What are the errors that could occur when children solve certain "Make an organized list" problem?
- What are the factors that would cause successful/unsuccessful behavior in solving certain "Make an organized list" problem?
- What are the difficulties to the adopted problem in this study?

Subjects and Problem

In this study, seven 6th grader children from a national university attachment elementary school in Japan are required to solve the problem below (this problem is referred as "Make-15-by-three-cards" after this).

"Make-15-by-three-cards"
There are cards of $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$ each.
You are required to choose any three cards that the sum of numbers on these cards is 15.
How many ways do you have to choose the cards?

It is important to make the subjects to solve problem that is suitable for the purpose in this study. It was found that most children in the elementary school could solve the usual combinatorial problems by pilot study and suggestion from teachers in the school. If a problem in the textbook or similar but a little more difficult problem is used in this study, the children could solve the problem easily and at the worst, they would solve it by using some routine procedure. In my opinion, such problem solving processes are not worth examining. Therefore, the above problem was adopted as a result of after some discussion with the teachers and other researcher of mathematics education.

Procedure of data collection

Each child (referred as subject) was asked to solve the problem on a sheet of paper in think-aloud manner. After the subjects have finished their solving activities (they were asked to tell the interviewer that they have finished solving the problem), they would then be interviewed about their problem solving processes. All sessions were recorded on tape-recorder and video-tape, and these data were analyzed.

3. Theoretical Framework

This study is designed to focus on the processes of subjects' problem solving. We want to analyze the usage of "Make an organized list" strategy, but that isn't sufficient and another factors are needed to be taken into consideration.

In this study, works of Schoenfeld (1985,1992) is used as the framework. Subjects' problem solving processes are analyzed in four categories of knowledge and behavior: *The knowledge base*, *Problem-solving strategies*, *Metacognition* and *Beliefs and affects* (for details of this categories, see Schoenfeld,1985,1992).

Taking this framework into consideration, we expect that the subjects need to know and have certain behavior in order to solve "Make-15-by-three-cards" more successfully. First, for *the knowledge base*, the subjects need to have basic computational skills and some sort of number sense that may help children to solve this problem. Of course, the subjects need to understand the meaning of this problem, for example, there is just one card each, the order of the card is not essential, etc. Second, for *problem-solving strategies*, the subjects need to use not only "Make an organized list" strategies, but also "Read the problem" strategy, "Look back" strategy or other strategies. Third, for *metacognition*, the subjects have to monitor and assess their problem solving processes and, if necessary, would change their solving processes. Such a metacognitive behavior is expected to play an important role in solving "Make-15-by-three-cards" because of the complexity of this problem. Fourth, with regards to *beliefs and affects*, it is not clear what kind of affective factor that would influence solving behavior of this problem, so it is interesting to identify such a phenomenon itself.

4. The Outlines of the Subjects' Solving Processes

Remark on terms and notation for describing solution

Before describing the subjects' solving processes, some terms and notation would be introduced. Assume that a child had constructed a list from top to bottom as in Figure 1. The term *component* is used to denote each sequence of number. In Figure 1, "9-5-1" is regarded as a component. Therefore, there are six components in Figure 1. Here, the form of components (9-5-1, 9+5+1, tree form or not,⋯) is not essential in this definition. We then denote the component, for example, "9-5-1" by [9,5,1]. In general, we denote component "a-b-c" by [a,b,c] and by this notation, it also means that the component was constructed a, b, c in this order.

9 − 5 − 1
9 − 4 − 2
9 − 3 − 3
8 − 6 − 1
8 − 5 − 2
8 − 4 − 3

Figure 1. An example of list

The term *list* is defined as a set of components. In analyzing subjects' problem solving processes, it is important to examine whether the construction process and the constructed list are systematic or not. In the case of the list as in Figure 1, for example, we can say that it is systematic, if we assume that the construction list of [9,5,1], [9,4,2], [9,3,3], [8,6,1], [8,5,2], [8,4,3] is in that order. For, the first three components that are constructed by using "9" as the first fixed number, then using "8" as the second fixed number. We express the results as *9-fixed-list* and *8-fixed-list.* We use symbol "x1" in order to clarify the first fixed number. And, we denote second and third chosen numbers for each x1 as "x2" and "x3" respectively. The notation above will be used flexibly to describe subjects' solving processes and hope this will not lead to any confusion.

Subject A's solving process

Subject A misunderstood the condition of choosing three cards for two cards. As a result, his answer was "2 ways", that is, $\boxed{6}$, $\boxed{9}$ and $\boxed{7}$, $\boxed{8}$. In the interview conducted after the solving process, the dialogue was as follows:

Interviewer: Don't you use other cards? You use only six, seven, eight, nine?

Subject A: No...,but three cards... Can I try again?

After this dialogue, Subject A re-solved the problem and made a list.

Subject B's solving process

Subject B didn't make any list and he calculated: $(9-1)\div 2$ and drew the conclusion, 4 ways. The dialogue in the interview was as follows:

Interviewer: What did you do at the beginning. I couldn't understand what you did.

Subject B: At the beginning, I misread this part (pointing at "choose any three cards" in the problem text), I have to choose three cards.

I : At that time, what did you do?

B: I thought it mentally. Then, I tried to find some pattern.

I : How did you try to find it?

B: The sum of these part is fifteen (enclosing the part " $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ *" in the problem text with a rectangle). But, it is impossible to do that in these part (enclosing the part "* $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$ *" and then "* $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ *" in the problem text.)*

(conform of what his meant)

I : How did you know that you misread?

B: I got lost, so I read the problem again, and I knew that I had to choose three cards.

(conform of the meaning of the sentences "9 − 1 = 8" and "8 ÷ 2 = 4" on the answer sheet)

Subject C's solving process

Subject C made 15 by using *two* numbers (the middle part in Figure 2) and he wrote "2 (ways)", but by repeatedly reading the problem, he noticed that the problem condition is "to use three cards" *without* any suggestion. He then explored the problem by making a list (the right side part in Figure 2). He constructed a 9-fixed-list and recognized that the components have to satisfy x2 ≠ x3. He didn't make, however, any other component (for example, a 8-fixed-list) and he thought that there were four components for each fixed number from 1 to 9. He also recognized [9,b,c] and [9,c,b] are essentially same. So, he calculated 4 × 9 ÷ 2 and drew a conclusion, 18 ways.

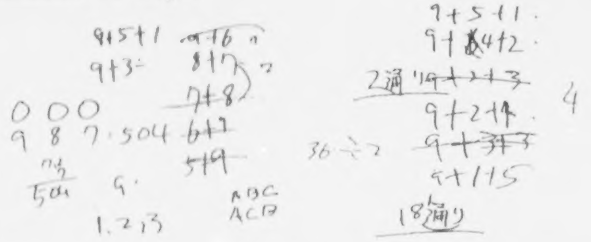
9+5+1
9+4+2
9+2+4
9+1+5
36÷2
18通り
4
ABC
ACB

Figure 2. Subject C's answer sheet

Subject D's solving process

Subject D constructed a list, and the process consisted of two phases. When she changed fixed number x1 from 1 to 6, for each x1, she calculated 15 − x1 and looked for pairs that satisfy x2 + x3 = 15 − x1, starting with x2 = 1, then she increased the value of x2 one by one. In these process, she wrote just components that satisfy 1 ≦ x3 ≦ 9 and x2 < x3. In the next phase, her ways of making components changed partly. She constructed those in this order: [7,3,5], [7,2,6] [8,3,4], [8,2,5], [7,1,7] (written under [7,2,6]) [8,1,6] (written under [8,2,5]), [9,1,5], [9,2,4]. As a result, a list showed below was constructed: [1,5,9], [1,6,8], [2,4,9], [2,5,8], [2,6,7], [3,3,9], [3,4,8], [3,5,7], [4,2,9], [4,3,8], [4,4,7], [4,5,6], [5,1,9], [5,2,8], [5,3,7], [5,4,6], [6,1,8], [6,2,7], [6,3,9] (error), [6,4,5], [7,3,5], [7,2,6], [7,1,7], [8,3,4], [8,2,5], [8,1,6], [9,1,5], [9,2,4] (her answer: 28 ways).

Subject E's solving process

Subject E's process of making a list was more sophisticated than subject D's one. She changed fixed number x1 from 1 to 6. For each x1, at first, she wrote all possible x2 that satisfied x1 < x2 , starting with the smallest number. After that,

she found x3 by computation; 15 − (x1 + x2) and she eliminated [a,b,c] if x3 didn't satisfy $1 \leqq x3 \leqq 9$ or [a,c,b] had already existed. As a result, the list indicated below was constructed: [1,5,9],[1,6,8],[2,4,9],[2,5,8],[2,6,7],[3,4,8],[3,5,7],[3,9,3],[4,6,5],[4,7,4],[4,8,3],[4,9,2],[5,6,4],[5,7,3],[5,8,2],[5,9,1],[6,7,2],[6,8,1] (her answer: 18 ways).
(Note: [4,5,6] and [4,6,5] were made in this order, but she eliminated [4,5,6])

Subject F's solving process

Subject F's process of making a list consisted of two phases. In first phase, he changed fixed number x1 from 9 to 7 one by one. For each x1, he chose x2 starting with 8 or 7, then he decreased the value of x2 one by one. He found x3 by the computation; 15 − (x1 + x2) except for the cases of x1=8 and x2=4,3,1 and he eliminated [a,b,c] if x3 didn't satisfy $1 \leqq x3 \leqq 9$. Unfortunately, he executed wrong subtractions in the cases of x1=7. At this point, his list was constructed as shown in Figure 3. Then, he continued to add [6,5,4],[6,4,5] and [6,3,6] to his list, and he eliminated components [6,4,5] and [6,3,6], saying that "*this* ([6,4,5]) *is the same as this* ([6,5,4]), *so it is eliminated, and 6, 3, 6..., but we don't have two six-cards, so it is eliminated.*" Then, he also eliminated [7,2,7]. After completing to make a 8-fixed-list, he eliminated [9,2,4];[9,1,5],[8,1,6],[8,2,5],[8,3,4],[7,3,6] and [7,4,5]. He returned to make 6-fixed-list and added [6,2,7] and [6,1,8] to his list. As a result, his list consisted of [9,5,1],[9,4,2],[9,3,3],[8,6,1],[8,5,2],[8,4,3],[7,6,3],[7,5,4],[7,1,8],[6,5,4] [6,2,7] and [6,1,8]. Because of computational error or for some other reasons, 9-fixed-list, 8-fixed-list, 7-fixed-list and 6-fixed-list all had three components! He made a judge that each fixed list has three components and calculated 9 × 3 (his answer: 27 ways).

951	861	763
942	52	54
933	4	45
924	3	36
915	2	27
	1	18

Figure 3. Subject F's list during solving

Subject G's solving process

Subject G's solving process could be divided as follows:

Phase 1: Construction a list, changing x1 as 9 → 1 → 8 → 7 → 6 → 5 → 4 → 3: As a result, she got her tentative answer "20 ways".

Phase 2: Check her list: She found four overlapping components and modified her answer as "16 ways".

Phase 3: Check her list again: She modified her answer as "15 ways".

Phase 4: Rewrite her list: She modified her answer as "6 ways" (Figure 4).

Subject G eliminated components when x3 doesn't satisfy $1 \leqq x3 \leqq 9$ or in the case that x1=x3 or x2=x3 as other subjects did. In addition to that, she succeeded to eliminate the essentially same components among different fixed-lists. She eliminated, for example, a component [1,5,9] which was included in 1-fixed-list, saying "*1 plus 5 equals 6, plus 9 equals 15. It's the seventh* (components), *but... Ah, since it includes 9... 1,5,9.*

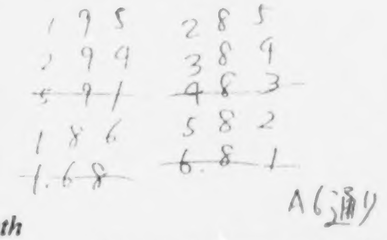
1 9 5
2 9 9
~~5 9 1~~
1 8 6
~~1 6 8~~
2 8 5
3 8 9
~~4 8 3~~
5 8 2
~~6 8 1~~
A(6通)

Figure 4. Subject G's list

It is the same as this(the component [1,9,5] which had been made earlier as a component of 9-fixed-list), *so, it is no use.*" She often said "*Since 7 has already been used...*" or "*It includes 8...*" and eliminated the components which included a number used as fixed number. On the other hand, her skill of constructing a list was not so refined like subject E. That made her list complicated, and unfortunately she missed to copy all components at phase 4.

5. Discussion

In this section, the errors caused by the subjects and the factors that would influence successful/unsuccessful solving behavior are discussed. The components of theoretical framework are used in discussion. Through these viewpoints, some suggestions for teaching and learning of problem solving that relates to the combinatorial problems would be elaborated.

5.1 *Understanding of the Problem*

Some Subjects couldn't understand some of the conditions that were imposed into the problem. Subject A, B and C misunderstood the condition of "three cards" for "two cards". We might say that it is just a careless mistake, but the solving processes of these subjects suggest that there is possibility to recover this kind of failure. Subject A misunderstood the condition, but by interviewer's indirect suggestion "*Don't you use other cards? ...*", he read again the problem, and noticed his misunderstanding. This episode suggests that if metacognition such as "*What is the given condition?*" (Hirabayashi & Shigematsu, 1988) or "*Have I used all the conditions?*" would have occurred, he might have realized this error. Even with two cards, this problem could also be a problem. Therefore, in this context, it might be difficult to expect the subject to re-read the problem for the reason that all conditions weren't used. Subject B and C, however, monitored their solutions and felt "it doesn't work" or "it is somewhat strange", and then, they re-read the problem and later reached the correct understanding of the condition.

As suggested, metacognition such as "*What is the given condition?*" and "*Have I used all conditions*?" could play some important role in successful problem solving (in "Make an organized list" problem, too). Therefore, we should make children and students to be aware of such kind of metacognition.

5.2 *Erroneous Multiplication and Possible Cause*

Subject C and F made erroneous multiplication. Subject C realized the number of components in 9-fixed-list correctly(two components), but he seemed to judge that any other fixed-list also has two components. Subject F's process of constructing a list had some merit, but his construction showed some error(for example, failure to eliminate [9,3,3] or computational mistakes like [7,6,3], [7,5,4] and [7,1,8]). It led to the appearance of the four number-fixed-lists that have all three components. However, it is not guarantee that all other fixed-lists have three components.

One possible explanation about these solving processes is that they have such a belief that "*There are some pattern in mathematical problem or its solution*" or "*A solution in a form of mathematical expression is better than another representations*". The latter belief, above all, might be the characteristic of

Japanese classroom lesson, since Japanese lesson usually emphasizes using mathematical expression in various phases than another representations.

Taking the problem type into consideration, they might have a belief "*The problem like combination or permutation could be solved by multiplication or division*". As far as their solution to the problem is concerned, there is no reason that the answer can be obtained by multiplication conducted by the subjects. "Look for a pattern" strategy itself is useful for solving a problem, but solvers should recognize that they have to think of the "reason" which supports the validity of the pattern before applying it. In addition, they should recognize that "the pattern" isn't a panacea for problem solving.

5.3 *Difficulties in "Make-15-by-three-cards"*

Subject D, E, F and G understood the conditions in the problem, and they constructed organized lists. During their solving processes, all ways that satisfy the problem conditions were appeared in all subjects' lists. This indicates that the subjects had the possibility of getting the correct answer (8 ways). Nevertheless, none of the subjects could get the correct answer.

These subjects' solutions are rather similar in the sense that all of them failed to eliminate the extra components (essentially same components), but we can notice the differences in their elimination processes. For the sake of analysis, the following viewpoints are set up:

viewpoint 1: Did they eliminate the components such that x1 = x2 or x2 = x3 ?
viewpoint 2: Did they eliminate the components [x1,b,c] if there also exist [x1,c,b]?
viewpoint 3: Did they eliminate the components such that x1 = x3 ?
viewpoint 4: Did they eliminate the components [a,b,c] if there exist for example [b,c,a] (their fixed number was different)?

Table 1 shows the subjects' elimination processes with respect to these viewpoints.

Table 1
Subjects' elimination processes

Subjects / Viewpoints	D	E	F	G
viewpoint 1	×	○	△	○
viewpoint 2	○	○	○	△
viewpoint 3	×	×	○	○
viewpoint 4	×	×	×	○

Note.
○ · · · eliminated perfectly (didn't write at all)
△ · · · eliminated partly
× · · · didn't eliminate

In Table 1 (the second row), we could notice that the elimination of overlapping components such as [x1,b,c] and [x1,c,b] is relatively easy for them (viewpoint 2). It just needs some knowledge of combination. Once the solver fixes the number of x1, the job becomes "two-dimensional". The subjects understood the condition that the order of number is not essential and they were familiar to the two dimensional combination.

Subject D didn't eliminate [3,3,9] and [4,4,7] (viewpoint 1). It was surprising that she didn't eliminate these components although she had already understood the "one-card-condition" (She made neither [1,7,7] after making [1,5,9] and [1,6,8] nor [3,6,6] after making [3,4,8] and [3,5,7].). The reason is that she had the wrong way to construct the list. When she constructed 3-fixed-list, for example, at first, she calculated 15 − 3 and then she looked for the pair that would make 12. At this point, she might pay attention to only the pairs. When she decided on [3,9], she should have checked whether the component [3,3,9], constructed with the fixed number 3 and [3,9] satisfies the "one-card-condition". However, she continued to make other pairs. Actually, this is one of the difficulties in this problem, i.e. the solver has to check the original problem conditions during or after "sub process" (in this case, making pairs). The two-dimensional problems or pure combinatorial problems do not have such difficulty. The difficulty in eliminating components such as that $x1 = x3$ in this problem (viewpoint 3) can be also explained in same way.

It is very difficult to eliminate components in the case of viewpoint 4. In addition to the difficulty mentioned above, for example, it is indeed difficult to recognize that the components such as [4,6,5] and [5,6,4] are the same. Furthermore, some essentially same components are far apart from each other, such as [1,6,8] and [6,8,1] in subject E's answer sheet. Even for an adult, it might be difficult to eliminate these extra components perfectly, which is another difficulty in this problem. One possible way to realize the successful elimination is to arrange the elements of the components in ascending or descending order (for example, as [6,8,1]→[1,6,8], [8,6,1]→[1,6,8]). But, unfortunately none of the subjects did it.

Taking all these difficulties into consideration, the "*construct full list, then eliminate extra*" method is not really a good way to solve "Make-15-by-three-cards". In general, It is important to write the components without omission and overlapping to solve this kind of problems. The "*construct full list, then eliminate extra*" method is valuable in the sense that it makes possible to avoid any omission at first. However, in solving problem like "Make-15-by-three-cards", this method is not adequate. Therefore, it would be necessary for children to have another alternative methods, for example, the "*constructing list, avoid used numbers*" method. As an example, after you have made 9-fixed-list up, you shouldn't use the number 9 again. This method has another value i.e. it could avoid any overlapping. However, it might be difficult for children to use this method, because it would impose high processing load on them. In this study, subject G used this method partly, and got the list that wasn't correct perfectly, but relatively a successful one. This suggests that it is possible for children at this age to acquire this method.

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EXPLAINING YOUR SOLUTION TO YOUNGER CHILDREN IN A WRITTEN ASSESSMENT TASK

Yoshinori Shimizu
Tokyo Gakugei University, Japan

Abstract: *This study explores the benefits of using "others" in a written assessment task as a technique for gathering rich data of students' performance in mathematics. A total of 64 eighth grade students in a public school were asked to work on a task that was presented in two different formats. In one version, students were asked to help a fifth grader who had been in trouble with a problem, while in the other version they were simply asked to solve the same problem and describe their solutions. Based on the analysis and comparison of students' responses to the task, the possibilities were discussed of assessing wider range of students' abilities and of raising learning opportunities by using "others" in a written assessment task.*

INTRODUCTION

The current emphasis in assessment practice in mathematics is moving away from assessing only students' knowledge of specific facts and isolated skills toward assessing students' full mathematical power (Ministry of Education, 1989; NCTM, 1995). In particular, teachers want to encourage students to think more deeply about the mathematics they are learning and they want to improve their own teaching by learning more about the range of their students' abilities (Stephens, Shimizu, Ueno, and Fujii, 1995; Shimizu & Lambdin, 1997). How can teachers become more familiar, through assessment, with the broad range of abilities, skills, and thinking of their students, and thereby more appropriately able to plan and modify their instruction?

In this paper, keeping the question in his mind, the author explores the benefits of using "others" in a written assessment task as a technique for gathering rich data of students' performance in mathematics. For the aim, students' responses to an assessment task, that was presented in two different formats but with the same mathematical content, were analyzed and compared.

The inclusion of "others" in a task is not a new idea. A supposed child who is in a problem situation often appear in mathematics textbooks, setting the context for presenting a task for learners. Written assessment tasks have been developed that are presented in the supposed situation of helping friend or person (Beesey et al., 1998). Further more, writing a letter to a supposed friend, who had been away from school for his illness, was asked to children to explore their affective aspects of learning mathematics (Ellerton, 1988).

On the other hand, from a general perspective, thinking can be seen as conversation with "generalized others" (Mead, 1934). Also, in a classroom setting classmates or the teacher will be a "significant other" (Bishop et al., 1996) who exert a strong influence on individuals. The importance of thinking "others" in learning mathematics has been explored, in the context of constructing mathematical definitions (Shimizu, 1996; 1997).

Communicating with "others", a supposed friend, for instance, students' thinking and their expressions seem to be promoted while they are working on a problem. Thus, from a teacher's perspective, using "others" in a written assessment task seems to have certain benefits for gathering informative data of students' performance in mathematics.

METHOD

Students: In this study, one of two versions of a task was given to 64 eighth grade students. The students were in two classes in a public junior high school in the suburbs of a city in Nagano prefecture.

The task: An assessment task, "Soccer Tournament" problem, adopted from Beesey et al. (1998), was given to the students with some modifications that had resulted in two different formats. In this paper, students' responses to each format will be analyzed and compared. Results of using another written assessment task were reported elsewhere (Shimizu, 1998).

Version A

You are planning a soccer tournament involving eight teams. How many games will be played in the tournament, if each team plays each other team once? Please describe the answer and how to work it out.

Version B

Kenta, a fifth grade student, has been put in charge of organizing a soccer tournament, involving eight teams. However, he has been perplexed with how to find the total of games, when each team plays each other team once. Please tell him the total of games and how to work it out.

Students are required to create and use diagrammatic, tabular and numeric representation of situation to aid in solving the problem. They may determine the number of possible pairs of teams and hence the number of games to be played.

The task was administered by the mathematics teacher of two classes, using approximately twenty minutes within a regular class session. Either the version A or the version B was given to the students in the following way. Each version was distributed alternatively to the students who sat in each row, so that the possible differences of mathematical ability within a class and between classes were set off. By this procedure, 33 students had worked on the version A, while 31 students had worked on the version B.

RESULTS

Students' responses were examined with respect to three distinct aspects: (a) whether their solutions were correct or not, (b) types of solution procedures, and (c) modes of explanations. These aspects will be described in this order and then discussed as a whole in the discussion section.

Students' Solutions

Table 1 shows the number of students, worked on each version of the task, who had gotten the correct or incorrect answer.

Table 1: Students' Solutions to Each Version of the Task

	Version A	Version B
Correct Answer	17 (51.5)[a]	19 (61.3)
Incorrect Answer	16 (48.5)	12 (38.7)
Total	33 (100)	31 (100)

[a] The number in parentheses is the percent.

As Table 1 shows, the students who had worked on the version B were more successful (61.3%) in solving the task than those on the version A.

Solution Procedures

A satisfactory answer to the task is likely to indicate thinking along the lines of 8 teams each playing 7 others (= 56) but team X playing Y is the same game as Y playing X (giving 28). If 28 is given as the answer and a reasonable explanation is provided, it is a quality response, while an answer of 56 or a systematic diagram that yields another incorrect answer may be considered as substantial progress, in spite of some minor flow. A student who is able to use a rule, possibly in algebraic notation, to generalize, will provide evidence of going well beyond the requirement of the task. In the group of students in this study, there was no such generalization.

The following categories were identified for classifying students' solution procedures. Students often used more than two procedures for finding the answer.

(I) Drawing a picture or diagram.

Students in this category simply linked the different teams visually, and concluded the possibility of games. The correct answer was obtained by providing a simple picture or a diagram. Incorrect answers gotten by the procedure in this category were obtained by an irrelevant attempt and/or an incomplete picture.

(II) Systematic counting.
This category includes those students who counted the total of games in a systematic manner, possibly by making a list of all games. In some cases, a number expression of addition, namely "7+6+5+4+3+2+1=28", was provided. An incorrect answer was obtained by counting systematically but in an incomplete or inefficient manner.

(III) Making a table.
This category includes those students who used a table to count the total of games. The incorrect answer that was obtained by using a table without considering the double counting falls into this category. In this case, the answer would be "56".

(IV) Identifying possible pairs.
This category includes those students who had indicated possible pairs of teams by thinking along the lines of 8 teams each playing 7 others (= 56) but team X playing Y is the same game as Y playing X (giving 28). An incorrect answer could be obtained by identifying possible pairs in the same way but by making a simple mistake in the execution of calculation, while the correct answer may be given by the expression "$8 \times 7 \div 2$". Another approach included in this category was based on such an idea that each team plays 4 others in one round, of four simultaneous games, and we will have seven rounds in total. In this case, the answer was provided with the expression "4×7", or "7×4".

(V) No explanation.
When only an answer but no work was provided, students' responses fall into this category.

Table 2: Solution Procedures

	Version A		Version B	
	Correct	Incorrect	Correct	Incorrect
(I) Drawing a picture or diagram	4	5	0	3
(II) Systematic counting	8	3	9	0
(III) Making a table	6	5	14	10
(IV) Identifying possible pairs	4	7	2	6
(V) No explanation	1	0	0	0

As Table 2 shows, solution procedures used by students were different in certain ways between two versions of the task. In particular, it should be noted that many of students worked on the version B made a table to explain their approach to the task (A: 25.6%, B: 54.5%), while many of those on version A used drawing (A: 20.9%, B: 6.8%). Also, within the version B, all the students who used systematic counting were successful.

One of the differences that deserve our attention here is the difference in students' explanations who had used a table. Actually, two groups in category III showed a significant difference. The numbers and percentages of students who explained how they could have found the answer by writing verbal expressions are shown in Table 3.

Table 3: Explanations by the students in Category III

	Version A	Version B
Table with words	5 (45.5)[a]	19 (79.2)
Table without words	6 (54.5)	5 (20.8)
Total	11 (100)	24 (100)

[a]*The number in parentheses is the percent.*

As Table 3 shows, the percentage of students who had provided verbal explanations for their solutions, as well as the table, was much higher on the version B (79.2%) than the version A (45.5%).

Modes of Students' Explanations

Students' responses were categorized from the viewpoint of whether they had or had not used visual explanations (pictures, diagrams, or tables), and/or visual explanations (explanations in words, phrases, or sentences). Students' responses were quite varied in their modes of explanations, when we classified them into "verbal only", "visual only", or a combination of them. Table 4 shows the classification in terms of these categories within each version.

Table 4: Modes of Students' Explanation

	Version A	Version B
Visual only	9 (27.3)[a]	5 (16.1)
Combination of Visual and Verbal	21 (63.6)	26 (83.9)
Verbal only	2 (6.1)	0 (0)
Not Provided	1 (3.0)	0 (0)
Total	33 (100)	31 (100)

[a]*The number in parentheses is the percent.*

The majority (83.9%) of students who had worked on the version B had explained their solutions both visually and verbally. On the other hand, 27.3 % of the students on the version A had provided only visual explanations.

DISCUSSION

As the results reported in the previous section show, there were significant differences in students' responses between the version A and B, although we should be careful to discuss the results obtained from a relatively small number of students. In the following section, we will discuss, within the limitation of number of students participated in the current study, the benefits of using "others" in a written assessment task for gathering rich data of students' performance and possible use of "others" in written assessment tasks.

First, students who had worked on the version B were more successful on the task than those on the version A. Also, solution procedures that students used to solve the task as a whole seemed to be more sophisticated with the students on the version B, when we look in particular at the categories of "drawing a picture or diagram" and "making a table". Further more, within the category of the "systematic counting", all of students who had worked on the version B were successful, while some students on the version A had failed to get the correct answer. It seems to be safe to say that the version B made students to work on the task more carefully than the version A did.

Secondly, the difference of modes of students' explanations between the version A and B is another point that deserves our attention with respect to the aim of the current study. The percentage of students who had provided verbal explanations to the version B was much higher than those on the version A. In terms of the modes of explanations, the majority of students who had worked on the version B had described their solution both visually and verbally, while a third of the students on the version A had provided only a visual or verbal explanation.

An interpretation of the results would be that we become more reflective by being put in the situation in which we have to tell something to other person than the situation in which we are simply asked to solve a problem. A closer look at some students' explanations supports this points. In some cases, for example, students had provided informal and colloquial expressions that suggest the students were aware of their own thinking, when they had tried to "teach" Kenta how he could have gotten the total of games.

"Telling" to a younger child is likely to raise a opportunity for the students to describe their solution in a "teaching mode". Actually, students who had worked on the version B seemed to be more "talkative" in their writing and their comments were in a broad range of descriptions. For those students who had made a table, the percentage of students who had provided verbal explanations for their solutions, as well as the table, was much higher on the version B (79.2%) than the version A (45.5%). Also, some students on the version B had made approach to Kenta by writing his name in their explanations.

In sum, the task that asks students to explain their idea to a supposed other, a younger child, in this case, seems to set the situation where students carefully try to communicate mathematics in various modes. From the teacher's perspective, using "others" in a written assessment task we can collect informative data of students' performance in mathematics.

Another issue that the results of current study suggests is the role of explaining to a supposed other as raising a learning opportunity. Although assessment is often used only for grading, it should enhance mathematics learning (NCTM, 1995). Students' use of multiple modes of explanations to the task that asks them to help a supposed other would raise an opportunity for learning the aspects of task from multiple perspective. This issue needs further explorations.

Inclusion of "others" in a written assessment task makes it possible for us to assess wider range of students' abilities than simply asking to solve a problem and to explain the solution. An effective format of using "others" in a written assessment task would be to show the situation in which a younger child is perplexed to be faced with the problem, like the version B in the current study. In a similar way, we can create a task that probes students' conceptions of certain topic or mathematical content, by showing other conceptions as being held by a supposed friend that seem to be in opposition to their conceptions. Another possible use of "others" in a written assessment task is presenting two or three ideas as being held by supposed others simultaneously, asking students to select one of them and to describe the reason why they choose it. By using this format, we can narrow the range of responses to be provided and elucidate students' thinking.

As was mentioned earlier, the results reported in the current study are based on the survey with a relatively small number of students. We need a further study with a larger group of students and with students in different grade levels.

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RECONSIDERING MATHEMATICAL VALIDATION IN THE CLASSROOM

Martin A. Simon, Penn State University

Recent mathematics reform efforts have embraced ideas such as "deductive reasoning is the means for determining the validity of a conjecture" and "students are responsible for validating their mathematical ideas." This paper suggests that a more complex consideration of these issues is necessary. In particular it raises questions about the following: How might students make use of inductive and deductive reasoning in order to convince themselves that the mathematical ideas that they have generated are valid? What is the appropriate role of the instructor in promoting student generation of mathematical ideas that are considered valid beyond the classroom community? How can the roles of students and instructor, with respect to mathematics validation, be understood by students?

Current efforts to reform mathematics education have promoted change in the roles of mathematics instructors and their students. (The use of the term "instructor," rather than "teacher," is used to be inclusive of those who teach post secondary, including teacher educators.) The most significant change has occurred in how new mathematical ideas are introduced into the classroom. In traditional classrooms, new ideas were generally imparted by the instructor; in reform-based classrooms students actively participate in the generation of ideas. One result of this change is that mathematical validity in the classroom has become problematic. Whereas in traditional classrooms, students assumed that the mathematics imparted by the instructor was valid, student generation of mathematics has introduced into class discussions mathematics of questionable validity. In this paper, I raise theoretical issues regarding the establishment of mathematical validity in the classroom. I use the term "theoretical" not to denote issues that are divorced from teaching practice, but rather to denote consideration of ways to think about practice.

Current reform efforts include engaging students in judging mathematical validity. Towards this end, instructors refrain from revealing their own judgements as to the validity of the mathematical solutions and assertions generated by the students. Such restraint on the instructor's part is seen as essential to promoting sincere engagement by students in determining mathematical validity. If the instructor were to rule on the mathematical validity of student work, students might devalue their active work and see the instructor's specification of the correct mathematics as the important part of their classroom experience.

The stimulus for my attention to the theoretical issues discussed herein has been reflection on a recent teaching experiment in which I was the mathematics instructor for a combined group of practicing and prospective teachers. Both the class discussions and interviews with participating teachers about their experience in the course served to prompt my re-examination of issues of validation and justification.

The teacher education classroom from which the data were taken was one in which there was a clear norm that students justify their assertions and that other students judge and critique the validity of the mathematical ideas presented. It was also well established that the instructor would not indicate his thoughts about the validity of ideas in question. Students often produced powerful deductive arguments to justify their solutions and ideas. I indicate these characteristics of the classroom to distinguish it from one in which students are just beginning to engage in mathematics in these ways. My purpose in this paper is to consider how we might think about validation in a classroom that has developed a stable practice of inquiry mathematics (Richards, 1991). In particular, I examine the contributions of students' inductive and deductive reasoning and the instructor's role in establishing mathematical validity. Further, I raise questions about the tensions in the instructor's role and the nature of *contrats didactiques* that can support inquiry mathematics in the classroom.

Conceptual Framework

Fundamental to current reform efforts in mathematics education is consideration of the mathematics classroom as a mathematics community (Voigt, 1995). The mathematics classroom community is intended to be a social structure in which authentic mathematical activity is practiced by the students. "Authentic" suggests that the activity of the classroom community members is consistent in significant ways with the activity of the community of mathematicians. It is an ongoing challenge for mathematics educators to understand the nature of mathematical activity and to consider how the classroom community can be structured to foster such activity. This challenge also includes determining the unique aspects of each community and, thus, the asymmetries of practice that are appropriate to preserve.

Both the mathematics classroom community and the community of mathematicians address issues of mathematical validity. However, the classroom community has the additional responsibility of constituting mathematical knowledge that is compatible with that of the community of mathematicians and with how mathematics is used in other communities outside of the classroom. The instructor serves as a bridge to these communities (Lampert, 1990).

The community of mathematicians has established mathematical proof as its recognized vehicle for establishing validity.

> The criteria which mathematicians appear to apply, consciously or unconsciously, are that the proof must proceed from specific and accepted premises, must present an argument that is not flawed, and must lead to a result which, even if unexpected, seems upon reflection to make sense in the context of other mathematical knowledge. (Hanna, 1990,p. 8)

Chazan (1990) stressed that the notion that mathematical proofs establish validity is problematic. In particular, proofs require interpretation and flaws may be detected at a later point in time. Thus, while the mathematics community has canons for establishing validity, the validity of any particular mathematics is never addressed with finality.

In the classroom, part of learning mathematics is learning to use accepted means of determining mathematical validity. Our recent teaching experiments have been guided by the frameworks described in the next section.

Hierarchy of Levels of Justification

"The definition-theorem-proof approach to mathematics has become almost the sole paradigm of mathematical exposition and advanced instruction" (Davis & Hersh, 1981, p. 306). However, researchers have observed that students, at a range of academic levels, seem to have other means for judging validity. Further, several of these researchers have postulated developmental hierarchies with respect to mathematical justification (Balacheff, 1987; Harel & Sowder, 1998; Simon & Blume, 1996, Van Dormolen, 1977). These hierarchies suggest a development from inductive (empirical) reasoning toward deductive reasoning and toward a greater level of generality. Simon and Blume, in describing data from traditionally-reared prospective teachers postulated a level preceding inductive reasoning, "appeals to external authority."

These developmental hierarchies are based on several observations:

1. Students who are beginning to engage in exploring mathematical situations, making conjectures, and determining the validity of their conjectures tend to verify their conjectures based on whether it works for the examples considered (Harel & Sowder, 1998).

2. Even when students use deductive reasoning at times, they often do not recognize that this approach is different from inductive reasoning, nor that the deductive treatment has a power not contained in the inductive (Simon & Blume, 1993).

3. Mathematical development, fostered by instructors, can lead to more regular and conscious use of deductive reasoning (Simon & Blume, 1996).

Further, these hierarchies are based on mathematics educators' assumptions that deductive justification is essential to mature mathematical activity (Hanna, 1996).

The Lived Experience of Mathematics in the Classroom

It is my experience that if I listen carefully to my students (whether children or adults), they can teach me about the lived experience of struggling with mathematical validation in the classroom. The first thing that I have learned is that the students' question is often "How do I know if this mathematical idea is correct?" "Correct" probably has two meanings (not necessarily articulated): "works for the situations that we are considering" and "is consistent with what others outside of this class believe" (e.g., the next mathematics instructor). The reader should note that this question of mathematical validity only arises when the mathematics is novel and challenging for the students. That is, the students' concerns are different when they are asked to prove an idea that they already know to be correct.

One incident from our recent teaching experiment with teachers and prospective teachers and one teacher's reflections on her experience in the teaching experiment were instrumental in causing me to re-examine issues of mathematical validation in the classroom. I will summarize the first and give excerpts of the data from the second.

The Fallibility of Deductive Justification

The students (teachers and prospective teachers) were working on a task that engaged them in re-inventing theorems for demonstrating congruent triangles. My goals as instructor included not only the geometry content knowledge involved, but also an understanding that a deductive argument is how mathematical validity is established. The students first identified SAS (two sides and the included angle) and offered the following deductive argument: if we have specified an angle measure and we mark off known distances (measures of the two sides) from the vertex of the angle along each of its rays, there is only one line segment (third side) that can connect these two marks (i.e. the endpoints of the first two sides). This argument was accepted by the students.

The discussion then turned to SSA (two sides and a non-included angle). Students produced what they considered to be an equally persuasive (and logical) argument that, if once again you begin with a given angle and you mark off the first side along one of the rays, there is only one point on the other ray to which a segment of a given length (the second side) can be drawn from the mark on the first ray. (When one draws the second side as equal to or longer than the first side, this is true. Perceiving the flaw in this argument involves drawing the right picture and thinking to swing the arc sufficiently far to identify the two intersections with the second ray.)

The students seemed to accept SSA as valid until Lori went to the board to demonstrate her small group's conclusion -- that SSA does not necessarily produce a unique triangle. The effect of Lori's demonstration was dramatic. Not only were the other students surprised and convinced by Lori's demonstration, but it caused them to realize that one could never be sure that a logical argument (deductive justification) was without flaws. Having earlier explored the fallibility of validation through trying examples (inductive reasoning), they were now acutely aware that deductive reasoning also does not produce a definitive verdict as to the validity of an idea.

Ivy's Reflections on Her Experience as a Student

Ivy, a sixth-grade teacher, was an active participant in the mathematics class for teachers described above. Ivy was interviewed about her experiences in the course during the unit on congruent triangles. We pick up the interview as she is discussing work that took place in her small group with respect to a construction problem. ("R" indicates the member of the research team who conducted the interview.)

I: But we never came to any definite conclusion. I never do though [in this class] unless we have a whole group discussion. . . . Even with four or five people in my group, I'm not comfortable with [only] that number of people agreeing. . . .

R: How many people does it take?

I: [The instructor], someone I perceive as knowledgeable about it. I don't know if I'd be -- I guess I'd be comfortable if the whole class was agreeing. . . I'd be comfortable for that class. I don't know that I'd take it out into the real world. [If the instructor tells] us, I'll actually be sure that I'm right. But I guess it's basically if . . . somebody who appears to me to be mathematical and knowledgeable about it said it was true.

R: Even if you had gone through a series of logical arguments like you were describing before?

[Ivy refers to the SSA incident described above.]

I: I did that on that triangle thing. And I totally missed the other little triangle that was created. It was completely rock hard clear in my mind, that triangle thing. I thought I was right, and I'm happy to share that. I'm confident enough to go in front to the board and show what my thinking was, but I'm anxious to see where somebody else thought because it's gonna come, there's something I easily could have missed. So no, I do not, even if I go through it with all those little clear things. I need, at this point in my life, somebody who or something, a book, something that appears authoritative that I would classify it as yes I definitely have it.

At first pass, one could interpret Ivy's remarks as a traditional view of mathematics learning, in which validation is determined by an authority (e.g., instructor or textbook). However, Ivy is not a traditional teacher. As a student, she engages in producing justifications to share with her peers. As a teacher, she often requires her students to justify their claims. Ivy's remarks can be interpreted as describing the experience of a student (albeit a practicing teacher) in a reform-based classroom.

Implications for Establishing Mathematical Validity in Classrooms

Reflecting on the data presented above, it becomes clear that mathematics educators must take seriously students' struggles with mathematical validity. How do students come to know that they know? In this context, I draw several conclusions and then initiate a conversation about *contrats didactiques*.

Conclusions

The role of deductive arguments in the classroom. In inquiry mathematics classrooms, deductive arguments (proof) are generated by students, not presented by instructors. As a result, the students come to understand through experience that there is always the possibility that their reasoning is flawed. Thus, while a deductive justification accepted by all of the students present can significantly augment their confidence in the validity of an idea, it may not result in a feeling of certainty. Thus, students might still seek further confirmation of validity. Students seeking further confirmation does not necessarily indicate a lack of understanding of deductive justification.

The role of inductive reasoning in the classroom. In the literature, inductive reasoning, trying examples, is often characterized as useful for generating conjectures (theorems). Further, students who have chosen to look at examples, even though they had access to a deductive justification, have been characterized by researchers as not understanding the role of deductive argument. It seems reasonable to consider that students might combine the two processes (inductive and deductive) in an attempt to increase their certainty. In the case where a deductive justification is generated early on, one might want to see the justified assertion work with some examples to increase confidence that the reasoning was appropriate. Mathematics educators might think about instruction as fostering competent use of both inductive and deductive reasoning and promoting understanding of the contributions that each makes to mathematical validation. (I also refer the reader to an earlier article, Simon, 1996, in which I argue for transformational reasoning as essential to a sense of knowing.)

The role of whole-class discussion. Ivy's experience invites us to consider the appropriateness of Ivy's dependence on the classroom community for increasing her certainty. The more people who consider the argument, the greater the likelihood that any flaws will be identified. This is not unlike the experience of a mathematician who has proven a new theorem, then asks colleagues to look at the work, and finally publishes it. The mathematician increases her/his sense of certainty about the work as the number of peers who have reviewed it increases.

The role of mathematical authority in the classroom.

One of the goals of inquiry mathematics in classrooms has been to promote self–reliance of students with respect to determining mathematical validity as opposed to their reliance on instructors and textbooks. Ivy's desire to have her mathematical assertions validated by an authority can be understood either as an immature perspective or as implying the following perspective. Given that there exists a community of mathematicians who have considered the mathematics in question and who have far greater knowledge of the mathematics and experience with mathematical justification, students cannot reach maximal confidence until a member of that community validates the mathematical work in question. Further, the presence in the classroom of a recognized representative of that community, the instructor, is a constant reminder of the possibility of arriving at greater confidence. Students may assume that the instructor knows whether the assertions under discussion are considered valid by the mathematical community and is aware of any flaws in the students' reasoning.

We can contrast and compare this experience to that of research mathematicians. Mathematicians working at the edge of knowledge in their fields of specialization generally do not have individuals with greater mathematical authority to validate their work. In this way the work of mathematicians is unlike the work of mathematics students in the classroom. However, in the case where a mathematician develops and proves a conjecture outside of her or his specific field of expertise, it is likely that s/he will seek the evaluation of an expert in the relevant field. It seems natural and appropriate to seek the evaluation of more knowledgeable persons when possible.

Towards What *Contrat Didactique*?

Let us consider classrooms in which the instructor monitors mathematical validity, refrains from indicating her/his judgements regarding validity, and finds ways to foster conceptual change when flawed mathematics is arrived at by students. The students seek to generate valid mathematics and to know when and if they have done so. However, even when convincing justifications are offered and consensus is reached, doubt may remain in the minds of students. They may wonder whether the mathematics that they have generated would be considered valid in mathematical communities beyond the classroom. This raises significant questions regarding the *contrat didactique* towards which mathematics educators might work.

From the instructor's perspective, s/he is using her/his knowledge of mathematics to monitor the mathematical understandings and practices of the students and to promote the development of valid powerful mathematical ideas. How do the students understand this situation? Can the instructor negotiate a shared understanding of the instructor's role as providing a safety net, taking some (usually indirect) action to promote change where mathematical ideas are seen as invalid? If achieved, such a shared understanding about the role of the instructor could have several implications.

1. The students would be aware that while they are exploring mathematical situations, making conjectures, and validating the conjectures, the instructor, who is facilitating the process and accepting all contributions, is also judging the validity of their contributions. What effect might the explicit knowledge that the instructor is monitoring validity have on the participation of students?
2. How do the students know when they have arrived at valid mathematics given that the instructor might still make an intervention to promote change in the current ideas? Do students come to know that their ideas have been approved by the instructor when s/he initiates a new mathematical topic? If so, what effect does this perceived pattern have on classroom mathematical activity?
3. In such classrooms students may work for extended periods with mathematical ideas that they feel relatively confident in, yet for which they still harbor doubt. This may imply a need for students to develop a different concept of what it means to know mathematics. Traditionally, knowing mathematics meant knowing what was certified as true by those in authority. In inquiry classrooms, it might be necessary for students to develop a notion of knowing mathematics as having ideas that are viable until they are shown to not be viable. Such a view needs to be integrated with an understanding of how the classroom functions to assure students of developing mathematics compatible with mathematics used outside of the classroom. Additionally, what effect would such a view of knowing have on evaluation that leads to grading?

Final Comment

Whereas recent mathematics reform efforts have embraced ideas such as "students validate their mathematical ideas" and "deductive reasoning is the means for determining the validity of a conjecture," this discussion suggests that it may be useful to adopt more complex notions. The human experience of developing confidence in the validity of a set of mathematical ideas can interweave inductive, deductive, and transformational reasoning with validation by experts. Classroom mathematics can be seen as a process within a process, that is, students doing mathematics in the context of an instructor monitoring their mathematical activity and promoting change in it. How mathematics students and their instructors can be helped to conceptualize this situation is a question for ongoing consideration.

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THE FORCED AUTONOMY OF MATHEMATICS TEACHERS

Jeppe Skott, the Royal Danish School of Educational Studies

Abstract: *Present developments in mathematics education provides the teacher with little assistance as far as recommendations for teaching methodology is concerned, and leaves him/her in an ironic situation of* forced autonomy. *Observations from three novice teachers' classrooms indicate how they cope with forced autonomy, and point to certain moments of their decision-making, the critical incidents of practice, that further complicate the task of the teacher by introducing additional motives of his/her activity beyond the teaching of mathematics. This may deplete the episodes of the intended mathematical contents, while the methodological and organisational approach is maintained. I suggest to use* intentional methodological discontinuities *as a means of avoiding this and as a relatively concrete methodological tool.*

Significant developments in mathematics education over the last decade have combined constructivist conceptions of learning with social perspectives on classroom interaction. In addition to these attempts to develop social constructivist understandings of mathematical learning in schools, the main emphasis in the conception of the contents of school mathematics has moved towards greater emphasis on mathematical processes, to some extent at the expense of the products that traditionally dominate the subject. Theoretically inspired by constructivism, interactionism and fallibilism, these developments conceive intended outcomes of mathematics teaching in terms of taken-as-shared mathematical concepts and procedures and of meta-mathematical conceptions of mathematics that reflect the processual emphases, for instance in the form of socio-mathematical norms (Yackel and Cobb, 1996). For brevity, I shall term these developments *the reform*[i].

The aim of this paper is to contribute to an understanding of the role of the teacher in mathematics classrooms inspired by this reform. I shall, then, address the issue of the possible implications for classroom teaching - if any - of the views of mathematics and of its learning, that influence the reform. I do so first by relating the reform to some of its theoretical sources of inspiration in order to outline some consequences as far as the teacher is concerned. Second, I elaborate empirically on this theoretical discussion by indicating the ways in which three novice teachers, who are all strongly inspired by the reform, deal with the complexities of the classroom interactions. Third, I introduce the notion of *intentional methodological discontinuities* as a recommendation for teachers when challenged by simultaneous existence of multiple motives of their activity.

The reform and its implications for teachers

A common feature of the mathematical and psychological sources of inspiration of the reform is the change in relative emphasis from mathematical products to processes. Lakatos' fallibilism and Davis' and Hersh' description of the mathematical experience are key sources of mathematical inspiration (e.g. Ball, 1988; Cobb, 1989, 1995; Cooney, Shealy, and Arvold, 1998; Ernest, 1991, 1998; Lampert 1990;

Skovsmose, 1990). When - in Lakatosian terms - mathematics is seen not as a set of eternal and indubitable truths, as foundationalism in all its forms would have it, but as chains of proofs and refutations (Lakatos, 1976), mathematical activity is a matter of becoming involved in these processes and not of acquiring a set of pre-existing concepts and procedures. And when - following Davis and Hersh (Hersh, 1998; Davis and Hersh, 1981) - mathematical concepts are seen not as objectively existing in an eternal Platonic realm, but as collective creations of human minds, and when these imaginary creations take on characteristics of their own, mathematical activity is the creation and modification of the objects and the search for their characteristics: it deals with the true facts about imaginary objects.

The psychological elements of the reform, based on constructivism and interactionism, emphasise both individual and collective processes (Bauersfeld, 1988; 1992; Cobb, 1989; Cobb, Yackel, Wood, 1992; Cobb, Boufi, McClain, Whitenack, 1997; Confrey, 1995; Voigt, 1996). Constructivism describes learning as the individual's successive approximations of his/her existing cognitive schemes through processes of assimilation and accommodation in response to cognitive disequilibria. In doing so it explicitly points to the need for the students to become involved in processes of active knowledge construction. The social elements of the reform relates both to the learning experience and to the intersubjective - or at least the taken-as-shared - existence of knowledge. This social perspective, then, maintains and expands the focus on individual construction, but does so by including social interaction as an inherent part of mathematical meaning-making, rather than as purely external factor with much the same role in subjective learning as the physical surroundings.

Consequently the student in the mathematics classroom is expected not only to come to grips with a set of predetermined concepts and procedures, but required to become involved in genuinely creative individual and collective processes of investigating, experimenting, generalising, naming and formalising; - and individually and collectively to develop mathematical concepts and procedures on the way.

A key issue for the realisation of these intentions is the role of the teacher within the reform. It is obvious, however, that the two elements of the reform described so far, those of the conception of mathematics and of learning, do not suffice as a basis for educational decision making in general, and for the development of appropriate teaching methods in particular. No philosophy of mathematics can by itself provide the necessary arguments for the contents of the school subject, and no epistemology or learning psychology can serve as the sole basis for the development of a teaching methodology. As Simon et al. (1999) point out with reference to the present reform:

> Using a theory such as constructivism to think about teaching involves a non-trivial adaptation from describing learning when it occurs to promoting learning where it might occur without an appropriate pedagogical intervention. (p. 203)

Further, it seems to be an essential feature of the reform that this 'non-trivial adaptation' must be carried out in the classroom by the teacher. It cannot be made de-contextually in more than very general terms and therefore does not in general call for the replacement of one set of teaching methods with another. It may deem very traditional types of teaching dominated entirely by teacher exposition of concepts and skills obsolete, but it cannot present a clear set of alternatives. Rather it requires the teacher to adopt a certain interpretative stance, to engage in reflective activity enabling him or her to flexibly use a wide range of different types of interaction with the students. This activity should be concerned with, for instance, (i) if individual students have the opportunity to become involved in processes mathematical knowledge construction; (ii) if concepts become taken-as-shared within the classroom and to an ever greater extent resemble the corresponding concepts of the broader community; and (iii) if also the meta-understandings and the acceptable forms of communication become taken-as-shared within the classroom, developing the class into a small community of mathematical practice in the process. It is a key element in this that the teacher is expected to respond instantaneously to contributions and questions of the students - individually and collectively - in ways that capitalise on the mathematical and meta-mathematical potential of these contributions.

Compared to the situation twenty years ago this points to a change from attempting to avoid teacher influence in mathematics classrooms to planning for his or her active contributions to curriculum enactment: We have come from teacher proof curricula to planned participation. An indication of this appears in the draft of *Standards 2000* (NCTM, 1998), where the requirements on the teacher is summed up like this:

> Curricular frameworks and guides, instructional materials, and lesson plans are only the first elements needed to help students learn important mathematics well. Teachers must balance purposeful, planned classroom teaching with the ongoing decision-making that can lead the teacher and the class into unanticipated territory from an effective mathematical and pedagogical knowledge base. (p. 33)

The teacher, then, is required to manoeuvre independently and autonomously in the classroom in order to provide the students with opportunities for mathematical learning and for developing taken-as-shared conceptions of mathematics as an ongoing human activity. This is a huge requirement, and from the teacher's perspective it may be seen, not as a longed for opportunity to play an important part in curricular decision-making, but as a new set of demands put on him or her as part of a new and slightly more advanced top-down strategy for educational development. From the teacher's perspective planned participation may be seen as an irony of *forced autonomy*. The notion of autonomy, sometimes understood as a paradigm case of an individual personality trait, is not to be understood as such in this context. Rather, the concept of forced autonomy claims, that the teacher is required to play a substantial role as a link between two sets of specific social spheres: On the one hand the macro-sphere of the institutionalised school mathematical priorities as described in the reform intentions and on the other the emerging micro-sphere of the mathematics classroom. Forced autonomy, then, refers to the demands put on the teacher as a result of his or her move to the centre stage of curriculum enactment.

Coping with forced autonomy

In order to understand of how teachers who are inspired by the reform may cope with the situation of forced autonomy, I followed three novice teachers with strongly reformist intentions of school mathematics for 2-3 weeks each. In a questionnaire before and in research interviews after their graduation from college, the three teachers used terminology like investigations and experimentation to describe the ways in which they envisaged the students' activity; they conceived mathematics as a way of approaching problems; and they presented their visions of teaching in terms that reflected intentions of being unobtrusively supportive. In short, the school mathematical images (SMIs) of these teachers, i.e. their espoused views of school mathematics and of its teaching and learning, were strongly inspired by the reform[ii].

All three teachers in the study engaged in classroom activities that clearly resembled aspects of their SMIs. For instance, in all three classrooms the students were encouraged to assign meaning to mathematical concepts by use of everyday language and multiple representations, before standard mathematical terminology was introduced. Also in all three cases the teachers and their students between them created atmospheres in which the students' contributions to the mathematical discourse in the form of posing problems, making conjectures and presenting justifications were clearly valued. And in general it appeared that the teachers responded to the students' use of everyday language and to their suggestions in ways that signalled the type of continuous reflective activity, that the reform calls for.

There were, however, also situations - the critical incidents of practice (CIPs) - when the enactment of the teachers' SMIs was challenged by the emergence of other motives of their activity beyond the teaching of mathematics (Skott 1999a, 1999b). These motives of teacher activity include, for instance, attempts to boost or maintain the teacher's professional authority, to manage the classroom, or to pursue broader educational aims (e.g. regarding students as children and not merely as students by supporting their self-confidence, by taking their family backgrounds into account, or by ensuring the position of individual students within the classroom community). When these additional motives emerged, the teachers sometimes got involved in classroom interactions that were apparently inconsistent with their SMIs, and they did not appear to engage in reflective activity with regard to individual learning and collective meaning-making. From this perspective they seemed to become engaged in oscillating practices, at times in line with the reform and at others hardly compatible with it. These episodes, however, should not be seen as examples of teacher inconsistencies, but as situations in which the multiple motives of the teacher's activity become subjectively incompatible, and in which the motive of facilitating mathematical learning is submerged, for instance, in broader educational aims.

In relation to the notion of forced autonomy this points to additional demands put on the teacher. When motives related to general educational priorities emerge, it requires the teacher to reflect instantaneously not only on elements of mathematics and its learning, but on a set of much broader considerations. The results from the study of the three teachers indicate that, when facing such demands the teachers sometimes successfully integrate or balance the different motives of their activity. The main characteristics of the episodes in which they do not succeed in doing so are that the methodological and organisational features of the situation are maintained, while its mathematical contents is discarded. To exemplify the point consider the following episode in which the teacher, Christopher, whose rhetoric and practice are both strongly informed by the reform, is teaching area and scales to a grade 6. The students are working in pairs, when two boys, Martin and Kaspar call Christopher.

Christopher: 1 centimetre is 2 metres. And there are 8 centimetres ... You still don't follow?
Kaspar: No.
Christopher: We'll start up here again [points to the task above] … 1 centimetre equals 100 centimetres in the real world … [awaits a reaction].
Martin: 1 centimetre is 1 metre in the real world.
Christopher: Yes, you can put it like that too. Why is that?
Martin: Well, [inaudible] that was just the way we calculated it.
Christopher: But that was because 1 metre is the same as 100 centimetres. Every time you have 1 centimetre then in reality it is 100 centimetres. And 100 centimetres that is the same as ...
Both students: 1 metre.
Christopher: 1 metre. That is why we say: 1 centimetre on the drawing is the same as 1 metre in the real world. Now you look at the one below [points to the original task].
Kaspar: That one, then, is 2 metres.
Christopher: Yes. That means that 1 centimetre is the same as 200 centimetres in the real world or 1 centimetre is the same as ...
Kaspar: 2 metres
Christopher: 2 metres. So that means ... and how many centimetres were there?
Martin: 8 centimetres.
Christopher: 8 centimetres. And how many metres is that in the real world? … [waits a second]. We can just start here [points to the drawing in Martin's book and moves his finger one centimetre at a time]. How many metres is this?
Martin: 2.
Christopher: And this?
Martin: 4
Christopher: Here?
Martin: 6
Christopher: Here?
Martin: 8 [Christopher speeds up the process, while Martin answers] 10, 12, ... eh 14, 16.
Kaspar: That is just the two-times table.
Christopher: So how many metres are there? When there are 8 centimetres, how many metres is that?
Kaspar: 16.
Christopher: 16. 2½ centimetres, how many metres is that?
Martin: That is then 4 .. 5 centimetres. 5 metres.
Christopher: 5 metres. Do you follow? [to Kaspar]
Kaspar: Yes.
Christopher: So, now we have 16 and 5. What are we going to do with those two numbers?
Kaspar: Add them.
Martin: Multiply.
Christopher: Multiply.
Kaspar: Oh well, yes.

> **Martin:** 16 by 5 [using the pocket calculator]. That is 80.
> **Christopher:** Now I want you to write it all down … [Christopher continues to work with them by telling exactly what to write].

I have previously analysed this episode as a CIP and argued that Christopher is as concerned with supporting the students' self-confidence and with managing the classroom as with teaching mathematics, and that in this sense he attempts to deal simultaneously with multiple motives of his activity (Skott, 1999a). This - at least in part - explains why the episode develops the way it does, as the intention of facilitating learning is dominated by other motives.

In the present context I shall view the episode from a different perspective, namely in connection with the notion of forced autonomy. The existence of additional motives of Christopher's activity - building students' self-confidence and of managing the classroom - points to further demands on him beyond those related to the reform as described above. He is required instantaneously to integrate the intention of facilitating the students' mathematical learning with the broader educational intentions. This indicates, that the notion of forced autonomy is not merely related to the reform of school mathematics: it involves issues of a more general educational nature and points to the need for the teacher to simultaneously address problems stemming from different domains. In short, such episodes present problems that are more complex than those of meeting the demands of the reform by themselves.

The episode above evolves so that the task is depleted of its mathematical contents (area and scales) and reduced to a question of multiplying 16 by 5 using a pocket calculator. This happens as a funnelling type of interaction develops, when Christopher - for reasons that have little to do with mathematical teaching and learning - narrows down the questions he asks, and the students refocus their attention from coming to grips with concepts and procedures related to area and scales to merely presenting an answer to the task.

However, while the intended mathematical emphases of the episode are discarded, the methodology and the organisation of the situation is maintained: The focus is on the textbook task and its solution throughout the episode; no additional teaching-learning materials are introduced to overcome the problems encountered by the students; and also the organisation - the small-group interaction with Kaspar and Martin at the table - is maintained.

In short, the interaction breaks with the intended mathematical contents of the episode, but maintains its methodological and organisational framework.

This seems to be the case for all three teachers in the study, when - in certain CIPs - the multiple motives of their activity appear to be subjectively incompatible, and their reformist SMIs are dominated by other motives of their activity. Analysing these episodes in retrospect, it appears that one way to address this problem is to do the opposite, i.e. to maintain the mathematical focus while discarding the methodological or organisational approach. To be specific, Christopher of course could have made a greater effort to relate the solution of the task to the previous one, that the students had solved successfully, although they had done so with little understanding. In this case he would have attempted to further exploit the potentials existing methodological framework. He may, however, also have initiated a whole class discussion on the notion of scales in which other students presented their solutions; he may have; or he may have asked Martin and Kaspar to leave the textbook for a while and make two different drawings of the classroom instead. In each of these cases he would break with the methodological or organisational framework of the situation and introduce *intentional methodological discontinuities* (IMDs) in order to keep the mathematical focus of the situation. Other interactions between the teachers and students seem to call for similar breaks with the existing methodological framework. I suggest to explore the potentials of the notion of IMDs further as a relatively concrete piece methodological advice to reformist mathematics teachers.

Summary and discussion

I initially claimed that current developments in mathematics education require the teacher to play an autonomous role in curriculum enactment, because of the increased emphasis on mathematical processes and of the inspirations from social constructivist conceptions of learning. The notion of *forced autonomy* was used to capture the irony of a situation in which the teacher is required to manoeuvre autonomously in relation to the mathematical contents and the students' learning.

Initially developed on the basis of developments in the theory of mathematics education the concept of forced autonomy was extended as a result of an empirical study of how three teachers incorporated broader educational priorities when participating in classroom interactions. This exemplifies that there are further dimensions to the demands put on the teacher than those inherent in the conceptions of mathematics and its learning.

The concept of CIPs was introduced to describe episodes in which multiple motives of the teacher's activity emerged as a result of attempts to relate simultaneously to the qualitatively very different demands that evolve in the classroom, for instance when attempts to address the issues raised by the reform appear incompatible with broader educational intentions. In these situations the intended mathematical focal points are, in effect, discarded, while the methodological and organisational framework is maintained. The notion of *intentional methodological discontinuities* is suggested as a means of maintaining the mathematical focus at the expense of the methodological framework. In one sense this adds yet another obligation to those already imposed on the teacher, and it does not present a methodological tool that relieves him or her of the responsibility of coping with forced autonomy. It does however, present a relative concretisation of the demands in forced autonomy or at least suggests a possible focal point for the teacher's reflection in the mathematics classroom.

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[i] The notion of *reform* is obviously problematic as it signals a general, one-dimensional development at both the theoretical and the practical level. In this context the term of *reform* is used merely as a short-hand for a broad set of theoretical and practical developments linking constructivist with interactionist perspectives on the classroom and emphasising mathematical processes as part of the contents.

[ii] For more comprehensive interpretations of the SMIs and classroom practices of one of the teachers, see Skott 1999a.

The Genesis of New Mathematical Knowledge as a Social Construction

Heinz Steinbring, Universität Dortmund, Germany

Abstract. *Does the development of mathematical knowledge in everyday teaching depend primarily on the individual insights of singular students or does it require a social discourse and a reflexive discussion? This problematique is an instance of the contrast between an »individual psychological perspective« that emphasises the autonomous, cognitive development of the learning subject, and a »collective perspective« that understands learning as a process of socialisation into a culture of teaching and learning. From an ongoing research project exemplary teaching episodes are analysed expressing how individual learning strategies (in seat work and partner work) and social-interactive constructions of knowledge (in common discussion with the whole class) favour different forms of an epistemological development of new mathematical knowledge. During phases of individual or partner work one can first of all observe linear, step-by-step procedures (algorithmic construction of knowledge); the common interactive phases of reflection offer opportunities for students to interpret the new knowledge from a conceptual-structural point of view (structural-systemic construction of knowledge).*

1 The Problem of New and Old Mathematical Knowledge

Mathematical knowledge is subject to a conflict in the relation between new and old knowledge: On the one hand every mathematical knowledge is logically consistent and hierarchically organised and can be deduced from given fundaments and, therefore, is not really new. On the other hand there are really still unknown and new mathematical insights, for instance by solving problems or by proving mathematical assumptions and theorems.

The example of mathematical proof explains this conflict paradigmatically. Rotman gives the following description: "… a proof is a logically correct series of implications… Proofs are arguments and, as Peirce forcefully pointed out, every argument has an underlying idea – what he called a leading principle which converts what would otherwise be merely an unexceptionable sequence of logical moves into an instrument of conviction… It is perfectly possible to follow a proof, in the more restricted, purely formal sense of giving assent to each logical step, without such an idea… Nonetheless a leading principle is always present… without which [proofs] fail to *be* proofs" (Rotman, 1988, pp. 14/5).

In an extensive historical and philosophical study, Jahnke analyses the contradiction between *development* and *justification* of knowledge and derives educational consequences for the learning of mathematics. "According to the self-image of science … »logic« and »intuition« are completely separated. .. The process of gaining knowledge is therefore essentially of an irrational character or at best to be explained as a psychological phenomenon, whereas a mathematical proof is only understood as a tautological chain of signs" (Jahnke 1978, p. 58/59). In the course of knowledge development new "ideal mathematical objects" are constructed that only permit to change the tautological justification on the basis of existing consistent knowledge into productive justification as regards mathematical content, a justification that argues from the future, i.e. it takes the new knowledge as its basis. "... the problem, to make students aware of the necessity of proving, shows that this only can happen with a look-ahead on more general points of view that make reasonable the possibility of an operative treatment of mathematical facts or of *idealised objects*" (Jahnke 1978, p. 251).

The distinction between the *logical structure* and the *(ideal) mathematical objects* becomes visible in interactions of every day mathematics teaching. The "logical consistency" is manifest in rules, laws of calculation and algorithmic procedures that are used more or less consistently as factual knowledge. In contrast, really new developed knowledge requires the interpretation of newly constructed mathematical relations. (cf. Steinbring 1999). This theoretical problem is an object of investigation in a research project ("Social and epistemological constraints of constructing new knowledge in the mathematics classroom"*), funded by the German Research Community, (DFG); cf. Steinbring 2000; Steinbring 1999); a central question is how children in primary grades are able to construct in an interactive way elementary new mathematical knowledge without having to use abstract notation or formal rules of production.

2 Theoretical Perspectives on Individual and Social Knowledge Construction

There is a controversial discussion whether new knowledge first of all emerges in form of individual contributions made by singular, competent students or whether it is the result of the social interactional context. The most prominent theoretical attempts within mathematics education trying to explain this contradiction are, on the one side, the social activity theory (according to Vygotsky) and the extension of the subjective constructivism to a social constructivism under an intersubjective perspective (cf. Lerman, 1996) and, on the other side, the radical constructivism (in its individualistic and interactionistic form). A common view is that in the subject dependent construction of new knowledge, both, the individual and the social environment are involved. But the "connection" between individual and social environment are differently judged.

As cultural tools for the acting subject, symbols have a central meaning for the relation between the individual and social position in the frame of activity theory. "Vygotsky's ... hypothesis is that all development of the individual comes about through sign mediation in activity" (Meira & Lerman, 1999 p. 3). But these cultural signs do not transport automatically new knowledge into the students' heads. "Following Vygotsky, cultural tools and sign systems are not carriers of meaning ...; that is, they do not carry their meaning to children. Instead, for any individual, cultural tools derive their meaning from his or her constructional activity" (Waschescio 1998, p. 234).

With regard to constructivism, Jörg Voigt makes the following distinction between an individualistic and an interactionistic perspective: "Individualism views learning as structured by the subject's attempts to resolve what the subject finds problematic in the world of her or his experience. Instead of asking how objective knowledge becomes internalised by a person or how subjective knowledge develops toward truth, interactionism and ethnomethodology study how intersubjectivity is achieved in the negotiation of meanings between persons" (Voigt 1998, p. 216). The main criticism of the constructivist positions first of all aim at the separation which, in the end, is assumed between an individual and a socially constructed meaning of knowledge or that – within radical constructivism – the social environment is seen as a mere formality and new knowledge is exclusively interpreted as a personal construction of an individual.

3 The Learning of New Knowledge as Collective Argumentation

The sociologist Max Miller (1986) makes the problem of the genesis of new knowledge the central question of what role the social, collective learning process has for the cognitive development of the individual. "From a theory of learning or development… one can legitimately expect, that it provides an answer to the question of how the *New can emerge* in development. … Every answer to this question … is subject to … the following criterion for validity: it has to be shown that the New in development presupposes the Old in development and at the same time exceeds systematically the Old, otherwise there cannot be a New or the New already is an Old, and then the concept of »learning« and »development« looses any sense" (Miller 1986, p. 18).

Then Miller puts three important questions for learning: "How for the individual … the validity of his or her already acquired (old) knowledge … can be shaken or relativized? How the individual can make new, his or her actual knowledge systematically exceeding experiences relevant for learning? And, how … can there be for the individual a compulsion to further develop his or her knowledge?" (Miller 1986, p. 18/9).

Genetic individualism cannot give answers to these questions. "Because within the paradigm of genetic individualism processes of learning are exclusively limited on mental processes of the single individual, also the constitution of experiences relevant for learning … can only be understood as a mental performance of the single, monological subject" (Miller 1986, p. 19). This theoretical position cannot explain how really new knowledge emerges. "In case that the learning subject already has to know somehow the New in the development for being able to get to know it first of all, then the New in development is already identical with the Old in development; …. The genetic individualism founders on a dissolution of the basic problems of a genetic epistemology" (Miller 19986, p. 20).

Individualistic orientations on "learning processes" concentrate – when taking a mathematical perspective – on consistent elements of knowledge which are already known and not really new, for example factual knowledge. Only collective processes make a potential development of new knowledge possible by contrasts, contradictions and re-interpretation. "What genetic interactionism substantially distinguishes from genetic individualism is the basic assumption that the collective and symbolically mediated application of mental capacities that are limited for the single individual might lead for the participating individuals to a process of experience constitution helping the single individual to solve the dialectic of knowledge and experience – of course not on the level of a theoretical (philosophical) reconstruction but on the level of an actual execution. Only within the social group and due to social processes of interaction between members of a group the single individual can make those experiences enabling fundamental steps of learning" (Miller 1986, p. 20/21).

Not every form of communication induces a learning process. "Only a type of discourse in which the principle goal is to find collective solutions to interindividual problems of coordination has a built-in capacity to release processes of collective learning. There is only one type of discourse that fulfils this condition: collective argumentation. Collective argumentations constitute the very basic method for jointly solving problems of interpersonal coordination" (Miller 1987, p. 231).

Collective argumentation cannot be reduced to the individual. ".. the method of collective argumentation cannot .. be described or explained by a reduction to a method of a mere individual argumentation. Also collective argumentations happen in the head of an involved subject, but the constitutive properties of a collective argumentation can only be understood adequately within a interaction theoretical frame" (Miller 1986, p. 25).

4 Differences in Interactive Knowledge Construction in Phases of Single and Partner Work or in Common Discussion – Analysis of Examplary Episodes

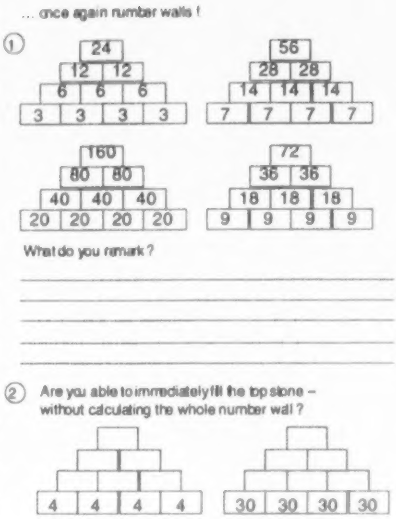
The detailed analysis of a number of teaching episodes in the mentioned research project (cf. Steinbring 2000) reveals that in the course of single and partner work students primarily concentrate on aspects of factual knowledge and re-construct consistent connections in already existing knowledge. With the perspective of unequivocal readings of mathematical signs learning is individualised in the sense of being seemingly simply the overtaking of ready, pre-fabricated mathematical knowledge and definite meanings. In contrast, the analysis shows that rather in common, social reflective discourses there are possibilities for open interpretations of mathematical signs and relational structures whose meaning still has to be established interactively in the course of discussion – dependent on the communicative style between teacher and students. The contrarieties in communicated readings of different symbols play an important condition for the construction of really new mathematical knowledge in reflective discussion.

4.1 Sonja and Julia Shorten her Calculation Process

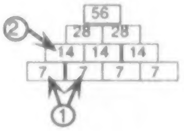
In the course of this teaching unit (grade 4) the students dealt with the learning environment of"number walls". During this lesson, the children worked on an exercise sheet with empty number walls in which the base row of stones was filled with four equal numbers. They had to calculate the missing numbers and to notice remarkable facts; another aim was to discover that the top number is the eightfold of a base number.

In the meantime, Julia and Sonja have filled the four number walls above; they want to report her observation to the teacher. First they simply remark: "There we have no longer really to calculate. There you have only three times".

In the following episode, they explain this remark more precisely; no doubt, they point at the exercise sheet but it is impossible to identify the exact number wall they refer to:

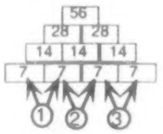

... once again number walls !
①
24
12 12
6 6 6
3 3 3 3
56
28 28
14 14 14
7 7 7 7
160
80 80
40 40 40
20 20 20 20
72
36 36
18 18 18
9 9 9 9
What do you remark?
②
Are you able to immediately fill the top stone – without calculating the whole number wall?
4 4 4 4
30 30 30 30

28 So One has always-, one only has to calculate three times, because these bo-, because, why this one gives that. ... One only has to calculate three times. [*points at some numbers on the sheet while she is explaining; then Sonja is interrupted by her neighbour Julia*]


56
28 28
14 14 14
7 7 7 7
②
①

29 Ju This, this plus this (1), this plus this (2) and this plus this (3). [*in this moment So and Ju speak simultaneously, they point both at exercise sheet in front of So*]

29a Ju Because, and when one has calculated these, these plu-, ehm, this plus this(1), gives this (2). And then one needs no longer to calculate these.[*meanwhile she points again at different places on the exercise sheet.*]


56
28 28
14 14 14
7 7 7 7
① ② ③

30 So And here the same [*points at different the exercise sheet.*]

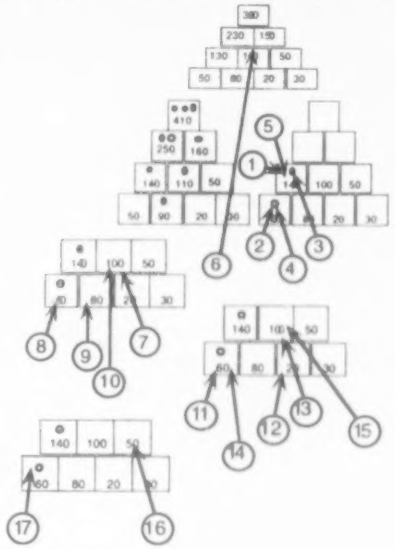
Later, Julia and Sonja report other striking observations: They have recognised that the calculated numbers in the fourth wall belong to the multiplication row of nine; moreover, the numbers double from one row to the next one above: "That's' always double, the double".

On the basis of their calculations Julia and Sonja construct her "abbreviation" and then they observe striking facts in these number walls. But they do not discover the "little trick" of an immediate calculation of the top number. The children's considerations remain on the level of a direct calculation and of visible numbers; for instance, no relations between the changes of the base stones and the resulting changes in the top stone are used (as has been done in the lesson before in some way). The teacher simply notes these explanations, partly she confirms them, but she remarks, that only one calculation task is necessary.

Summarising, one can conclude that the children in this episode do not construct a really new knowledge relation but they rather re-construct the interpretation of mathematical signs on the basis of already known solid knowledge and then they are able to abbreviate the calculation process – because of the observed repetitions; also the mentioning of the "ninth-row" and that there is "always the double" are forms of a re-construction of arithmetically consistent knowledge. The children reconstruct the uniquely fixed and pre-fabricated signifieds for the mathematical signs; for this purpose they use the mathematical sign–tools in the familiar manner.

4.2 Matthi Explains the Increase of the First Stone in the Second Row

This episode is part of the same teaching unit on number walls. Now, in the course of reflective discussion, an explanation is expected what is the change of the top stone dependent on the change by "10" of one of the base stones. Just before, the effect of the increase of one of the middle base stones has been discussed; now the relation between the left border stone and the top stone is in question. The student Matthi is asked to explain why the left stone in the second row of the wall has increased by "10", dependent on the change of the left base stone. Matthi goes to the blackboard and, first, he wants to calculate the next number row; but the teacher interrupts and asks for an explaining argumentation.



198 T # No. Matthi, please wait a moment? Did you notice something? #

199 Ma # Ehm, here (1) also is 10 more (2, chip)

20 Ma .... because also here is 10 more. [*places a chip* (3)] Because here, too, because it's 10 more here, (4, chip), here is 10 more (5) than there 10 more (6). Here then is the same (7) [*and then several times alternately* (8), (9)], because one cannot this here somehow plus that (10). One does not get with this here (*alternately* (11), (12)) then there (13), (14), or so. That one gets this. (15). And here (16) it's also the same, because this is on the outside (17).

Obviously, the student Matthi comes with the intention to the blackboard to calculate the numbers in the next row of the wall with the consistent rules. But the teacher's demand forces him to develop spontaneously an explanation for the increase of the according stone. Then Matthi constructs a new mathematical sign; he uses neither concrete numbers nor more general descriptions for the positions of numbers, but he always points at certain places in the number wall: This way of pointing is his personal way of expressing the mathematical signs he is constructing; he mainly bases his communication on signs produced in a deictic manner. He argues indirectly: If the sixth stone should increase then the first base stone must contribute to the corresponding addition; but this increased stone cannot be a term in the sum with the second nor with the third base stone, for producing the middle stone of the second row.

Matthi constructs a new mathematical sign that is not yet known or familiar. In the course of common interaction the students actively have to interpret this sign. Further, the construction is not simply only his very personal, individual construction; especially the discussion episode, just before, on the increase of numbers in the second row induced by an increase of a middle stone obviously had important influences on the way of how Matthi constructed his argument.

5 The Genesis of New Mathematical Knowledge: Individual or Social Learning?

In an exemplary way, the discussed episodes make visible two forms of interaction, on the one hand, the orientation on the mediation of consistent factual knowledge, on the other hand, the potential construction of new conceptual relations. In the context of consistent arithmetical rules, Julia and Sonja reduce the number of calculation steps to three. No really new knowledge is constructed, but while observing repetitions of calculations, the procedure is abbreviated. The knowledge here is derived from existing properties immediately.

Behind Matthi's indirect justification there is a construction of really new knowledge relations: The increased left stone in the base row is never linked with any other base number and therefore it is impossible to increase the stones except the left one in the second row. This argument cannot be deduced in this moment from existing factual knowledge. The question »What is a sufficient reason for looking at these – impossible – arithmetical relations?« requires to be aware of an "idea behind the logical structure".

At first sight, the constructions of knowledge always seems to be the result of personal insights of individual students. Of course, during processes of communication always some participant introduces his or her contribution and, reversal, the communicative process depends on these personal contributions. But this is not sufficient to deduce that the construction of new knowledge is exclusively an individual process. The obligation for constructing new mathematical signs and relations in the end is an external cause "imposed" to the individual by a contradiction, a conflict or a contrast in the course of social interaction – or also in a personal struggle with a contradiction in the existing knowledge structure.

The proposal made by Julia and Sonja rather shows a linear, step-by-step, algorithmic knowledge construction in the context of existing factual knowledge. The episode where Matthi develops his argument, exemplifies the interactive manner of a construction of new mathematical knowledge. The requirement to give a justification forces Matthi to exceed his own knowledge and to make "experiences relevant for learning".

In the course of every day, traditional mathematics teaching the requirement that students develop justifications and arguments plays a minor role; when students are more strongly asked to give reasons for mathematical insights – as it was the case in the teaching units observed and analysed in the mentioned research project – then one can recognise that especially phases with joint reflection and discussion offer opportunities to interactively construct really new mathematical knowledge. Social, collective communication is a particular basis for young students in primary grades to question their actual available, individual knowledge archive and to further develop their knowledge substantially. In authentic mathematical communication and argumentation the meaning of mathematical signs must not be predetermined and fixed, but the possibility of an intentionally open meaning of signs is the inevitable fundament for a collective learning process in which only mathematical meaning for these signs of new knowledge can develop interactively.

Acknowledgement

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MODALITIES OF STUDENTS' INTERNAL FRAMES OF REFERENCE IN LEARNING SCHOOL MATHEMATICS

Hitoshi Takahashi
Joetsu University of Education

The purpose of this research was to define some modalities of students' whole inner world and to explore some of the characteristics of these modalities. This research was based on Polanyi's theory of knowledge. Four participants were interviewed two times in grade six and two times in grade seven. Eight modalities of the participants' inner worlds, which make up their internal frames of reference, were found. They were their view of mathematics, attitude/affection towards mathematics, objectifying knowledge, influence of significant others, ways of learning, relating mathematics with daily life, relating mathematics with other subjects, and formalized mathematical knowledge. Although some of the modalities could change, a few were more coherent. This had an effect on the internal frames of reference. Some modalities were related to each other while other modalities exerted an influence on others. These modalities that make up the internal frames of reference depended upon the individual and sometimes directed knowing mathematics.

Introduction

A students identity grows when they develop their inner world through learning mathematics. The inner world of a student consists of many characteristics including attitudes, emotions, images, and metaphors. These characteristics are the foundation from which a student understands and appreciates mathematics.

In mathematics education perceptions about the nature of humanity and mathematics are the main elements. With this in mind this research is based on the theories of Polanyi (1958, 1966) and Polanyi & Prosch (1975). Polanyi (1958, 1966) proposed that scientific knowledge doesn't exist as an impersonal universally established and objective knowledge but is shared and developed by all types of scientists which form a society like chains of overlapping neighborhoods. Mathematical knowledge of a society also consists of relationships between personal knowledge and includes a tacit dimension. The word personal doesn't indicate self-righteousness or isolation in mathematics but includes the tacit dimension which tries to know actively through intellectual passion, images, and the belief in the existence of mathematical answers to problems.

Knowing depends on a conceptual framework which either assimilates new experiences or ideas, or adapts to them (Polanyi, 1958). The framework includes a tacit dimension which supports and fosters the activity of the framework. This tacit dimension also produces an integration of meanings which Polanyi & Prosch (1975) called metaphors. Polanyi & Prosch (1975) argued that when tenor and vehicle are integrated with the tacit dimension, which they refer to as ourselves, metaphorical relationships and new meanings are created embodying characteristics of the tacit dimension. Although their argument was about poetry their theory can be applied to mathematics. However knowing mathematics depends on the students basic knowledge because it is difficult to distinguish between the tenor and the vehicle.

Recently some research (i.e., Byers, 1999) has been concerned with ambiguity in mathematics. Other research (Chapman, 1997; Lakoff & Núñez, 1997; Presmeg, 1992, 1997; Sfard, 1994, 1997; Whitin & Whitin, 1996) has explored metaphorical thinking and also dealt with ambiguity. Their idea of metaphor was based on Lakoff & Johnson (1980) and Lakoff (1987) who proposed that mathematical concepts are constructed metaphorically so ambiguity is inherent in mathematical concepts.

This ambiguity is overlapped by the tacit dimension as described by Polanyi (1958, 1966) therefore their research has some common view points with Tanaka (1994) and Takahashi (1996a, 1996b, 1998a, 1998b) whose theoretical base was Polanyi (1958, 1966). Tanaka (1994) interviewed four seventh graders to investigate their connotative meanings of mathematical concepts, and found five modalities used in constructing connotative meanings. They were attitude/affection towards mathematics, daily life knowledge, images/metaphors, metacognition and previous learning experiences.

Research by Minato and his colleagues (e.g., Minato & Kamada, 1996, 1997) was concerned with the inner world as a whole and explored attitudes towards mathematics as connotative meanings not just denotative. Takahashi, Minato & Honma (1993) approached the theme by exploring elementary school students understanding of word problems which had different formats and situations. Unfortunately the results were too general because the research method relied solely on statistics.

In order to explain the process by which a student integrates their inner world to know mathematics Takahashi (1996a) extended Polanyi & Prosch's theory of metaphors (Polanyi & Prosch, 1975). According to Takahashi (1996a) students know mathematics more or less metaphorically. Takahashi (1996b) found that a seventh grader knew numbers metaphorically by comparing positive numbers and negative numbers to the trunk and roots of a tree. Takahashi (1998a) found that some coherent characteristics existed that fostered knowing three mathematical concepts. Takahashi (1998b) found that six graders had different views of mathematics, and that internal mathematics corresponded to positive learning activities while external mathematics corresponded to passive learning activities.

The purpose of this research was to define some modalities of students' whole inner world and to explore some of the characteristics of these modalities.

Methodology

The research method was interviews based on Moustakas' counseling theory (Moustakas, 1990). Moustakas (1990) proposed research method to explore the personal inner world is based on Polanyi's theory (1958, 1966). With this method the interviewer shares the interviewees inner world and seeks to obtain qualitative depictions of the persons experiences such as situations, events, conversations, relationships, feelings, thoughts, values, and beliefs (Moustakas, 1990). Moustakas phenomenological standpoint is compatible with this research because of the approach from many sides.

The interview method used was free association which is a counseling method. This kind of flexible interview approach had already been practiced by Cooney (1985) and Chapman (1997) to explore teachers inner worlds. Takahashi (1996a, 1996b, 1998a, 1998b) adapted Moustakas' counseling method to explore students inner world and obtained reliable results.

There were four students (male: Shuta, Atsushi; female: Yukiko, Haruko) all of whom were interviewed four times. The students were chosen as the result of a selection process using classroom participation, teacher recommendation, and a series of introductory interviews. The official interviews lasted from 40 to 70 minutes. The dates of the interviews and the grades participants were in were as follows.

Interviews	Date		Grade
First	Jul. -Aug.	1998	sixth
Second	Mar. -Apr.	1999	sixth
Third	Jul.	1999	seventh
Fourth	Dec. -Jan.	1999	seventh

During the second and fourth set of interviews a questionnaire was used in order to evoke the participants learning activity. The first questionnaire had two classroom-like situations. The first was to develop a formula for a parallelograms area and then to perform a multiplication of fractions. In the second questionnaire the problem was to solve a first degree equation and to draw a perpendicular line. During both of the questionnaires the students were requested to propose their own idea of solving the problems and to explain how they would propose their idea to their classmates. The purpose was to gain insights to their learning activity.

For the interviews a standard set of questions was used as a starting point. Further questions were dependent upon the participants answers.

Results

From the interviews eight modalities of the participants' inner worlds were found. They were their view of mathematics, attitude/affection towards mathematics, objectifying knowledge, influence of significant others, ways of learning, relating mathematics with daily life, relating mathematics with other subjects, and formalized mathematical knowledge. Some of these eight modalities were the same as the five connotative meanings found by Tanaka (1994). In this research I changed metacognition to objectifying knowledge to emphasis the character of objectifying. Metaphorical knowing was related with all the modalities, because its activity was included in the tacit dimension, so I didn't make it a modality.

Outlines of the four participants' modalities are as follows. Not every participant revealed all eight modalities. Two modalities, relating mathematics with daily life and formalized mathematical knowledge, aren't included in the results due to their obvious nature. They are referred to in the discussion when necessary.

Shuta's case

View of mathematics In the first and second interviews Shuta's view of mathematics was that it was practical and useful to their lives for shopping and consumption tax, etc. During the first interview he revealed the view that elementary mathematics had rules. In the second and fourth interviews Shuta expressed that calculating geometric areas wasn't useful because it was enough for him to get the actual sensation walking around the area. Also in the second interview Shuta stated that elementary mathematics was necessary for humans to develop and that learning was necessary for his life after growing up.

Attitude/affection toward mathematics Shuta displayed a good attitude throughout all four interviews. He described elementary mathematics and its class as enjoyable. In the second interview he said that it was interesting to develop his ideas from rules. In the fourth interview he stated that he enjoyed when his opinions were accepted by others and that learning numbers was more enjoyable than geometry because the concepts of geometry were natural for him.

Influence of significant others In the second interview Shuta said his good marks were the result of his father's advice.

Ways of learning In the first interview Shuta described that he worked at a problem until he could discover the rules. During the first and second interviews Shuta stated that correctness, short time, and brief expression were important in problem solving. In the fourth interview he revealed his opinion that equations are like balances.

Relating mathematics with other subjects In the second and fourth interviews Shuta connected the knowledge of natural numbers to historical or geographical knowledge such as periods of history or the populations of cities.

Yukiko's Case

View of mathematics In the first, second, and third interviews Yukiko described that elementary mathematics was practical and useful for measuring sizes of desks and classrooms and for calculating the birth of the universe. In the second and fourth interviews Yukiko stated that originality was important in the problem solving process in elementary mathematics and mathematics while the answers were already decided or fixed. She emphasized the importance of originality in the second interview stating that solving time was based on the individual and that formulas should have feeling for the person. She mentioned the importance of problem solving processes in the third interview. She made a distinction between elementary mathematics and mathematics in the fourth interview stating that elementary mathematics was practical and useful while mathematics was for enhancing her level of knowledge.

Attitude/affection towards mathematics During the interviews Yukiko had a good attitude/affection towards elementary mathematics. However when she started mathematics her attitude/affection started to became ambivalent. In the fourth interview she said that when teachers only explain things in class that the class was boring.

Objectifying knowledge Yukiko revealed mysterious feelings about triangles in the second and fourth interviews where she described some new geometric concepts as "flying into her head."

Inference of significant others In the second interview Yukiko compared the cyclical idea that some of her classmates would become elementary school teachers to the process of deriving geometric shapes from triangles.

Ways of learning In the second and fourth interviews Yukiko explained that she wouldn't propose an idea until she was able to explain the problem solving process using her own original ideas.

Relating mathematics with other subjects In the second interview Yukiko compared the derivation of geometric shapes from triangles with biological evolution.

Atsushi's Case

View of mathematics During the four interviews Atsushi described about the practicality and usefulness of elementary mathematics. In the first interview Atsushi said that numbers were responsible for the computer age and that elementary mathematics was a common language around the world although different symbols might be used because many countries had original numerals. During the second, third, and fourth interviews he described the necessity of mathematics for some occupations. Also in the second interview he described elementary mathematics as useful to him because it promoted his thinking ability. In the third and fourth interviews Atsushi described that the reason for learning mathematics was to prepare him for his future.

Attitude/affection towards mathematics Atsushi's attitude was good throughout the interviews but he revealed a preference for geometry over calculations because he enjoyed assembling plastic toy model kits and wasn't good at performing basic calculations.

Objectifying knowledge In the fourth interview Atsushi said that sometimes he couldn't keep up with the class because he felt that the numbers simply "ran around in his head."

Influence of significant others In the fourth interview Atsushi revealed that his preference for geometry was due to his father's hobby with motorcycles. His father often read design manuals so he could repair his motorcycles. Atsushi stated his interest in assembling plastic toy model kits stemmed from his father's hobby.

Ways of learning Atsushi stated in the second and fourth interviews that everybody could solve problems using their own ideas but in the case of calculations speed was also important.

Relating mathematics with other subjects Atsushi described relationships between mathematics knowledge and natural science in the second, third, and fourth interviews.

Haruko's Case

View of mathematics In the second and fourth interviews Haruko stated that elementary mathematics wasn't useful for her and stated that elementary mathematics knowledge was only useful for elementary mathematics. During the second interview she also said that elementary mathematics was necessary for humans because of our economic society. Throughout the interviews Haruko said that learning elementary mathematics and mathematics was only memorization and that she only did it for the sake of her future .

Attitude/affection towards mathematics Throughout the four interviews Haruko mentioned that elementary mathematics and mathematics was difficult, dull, and that she disliked it. During the second interview she said that if she could understand the meanings of elementary mathematics that she maybe would gradually learn to like it.

Ways of learning During all four interviews Haruko mentioned memorization as her only way of learning.

Relating mathematics with other subjects In the fourth interview Haruko said that learning expressions was useful for calculating density in natural science.

Discussion

The four participants were in the same classroom both in grade six and grade seven however their inner worlds revealed many differences when judged using the eight modalities revealed in this paper. Considering the participants personal learning history it is natural for differences to exist, but from the point of view of this research the tacit dimension (Polanyi, 1958, 1966) is the cause of the differences.

Through the participants internal frames of reference I could explore their inner world. Although some of the modalities changed between interviews, others remained consistent. Shuta, Yukiko, and Atsushi all had the consistent view that elementary mathematics was practical and useful for our lives. However Haruko had a consistent negative view and didn't think that elementary mathematics was very practical or useful. The positive view might reflect the nature of teaching materials and practices. But in Haruko's case other modalities could have influenced her view of elementary mathematics.

Consistency was found not only in views of mathematics but also in attitude/affection, ways of learning, relating mathematics with daily life, relating mathematics with other subjects, and formalized mathematical knowledge. Atsushi's and Shuta's attitude/affection remained positively consistent while Haruko's attitude/affection remained negatively consistent. The only change in attitude/affection was Yukiko's, which went from positive to ambivalent. Shuta's, Yukiko's, and Atsushi's ways of learning remained consistent. Unfortunately Haruko never revealed her way of learning because she didn't want to interact with other students.

There were relationships between some of the modalities. Some modalities interacted with each other positively and directed the participants towards knowing and learning activities. In the first interview Shuta's view of mathematics was that elementary mathematics has rules. This view interacted with his way of learning because he would work at a problem until he discovered the rules governing the problem. The view also disguised itself as an attitude/affection towards mathematics, because it was interesting for him to develop his idea from the rules. Another basic positive relationship was in Atsushi's case where he revealed his attitude/affection and view of mathematics was influenced by his father's motorcycle hobby.

A positive multi-relationship between modalities was revealed during Yukiko's second interview. The relationship was between objectifying knowledge (learning geometry), relating mathematics with another subject (biological evolution), and view of mathematics (the cyclical idea that someday some students will teach mathematics). Her learning of geometry started with triangles. To her triangles was the basic shape from which all other shapes were derived. She revealed that this idea was inspired by her knowledge of biological evolution. To her these are both cycles. She was able to relate these cycles to the cyclical idea of some students becoming teachers.

After she revealed these metaphoric relationships, she explained why she likes triangles. ``I like the formula the base multiplied by the height divided by 2 ..., I believe I will never forget it ..., the reason was that a teachers explanation was enjoyable and the image became clear ...". This is an excellent example of a teachers effect on a students inner world through the influence of significant others.

The relationship between modalities that interact with each other is subject to degrees of change. In the second interview Yukiko revealed a relationship between her view of mathematics and way of learning. She believed that original ideas are important but the answer is decided and that she would not negotiate with classmates until she had her own original thought. However by the fourth interview the relationship had changed. Her view of mathematics was that even though original ideas are important having the correct answer was becoming more important but she still would not negotiate without original thought.

Conclusion

In this research I found a lot of relationships among modalities. These relationships were, coherent or not, participants in the internal frames of reference, and influenced learning and knowing elementary mathematics and mathematics.

The internal frames of references are an important part of the students' inner world which has a direct influence on learning school mathematics. Usually we can recognize differences in knowing mathematics, but students develop their inner world through knowing mathematics and mathematics education. We should realize that a student's inner world has many sides and should work towards enriching it through mathematics education.

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THE EFFECTS OF METACOGNITIVE TRAINING IN MATHEMATICAL WORD PROBLEM SOLVING IN A COMPUTER ENVIRONMENT

Su-Kwang Teong, John Threlfall and John Monaghan

University of Leeds

Leeds, LS2 9JT, UK

This study demonstrates that explicit metacognitive training can increase students' mathematical problem solving performance in a computer environment. Twelve-year-old Singaporean students in collaborative pairs were assigned to three word problem solving groups. The first group received explicit metacognitive training before word problem solving with WordMath[1] (MAC); the second group undertook word problem solving with WordMath (AC); and students from the third group were taught word problem solving with paper and pencil (TC). Empirical results from the experimental and case study designs revealed that MAC students outperformed AC and TC students on ability to solve word problems and that MAC students elicited better-regulated metacognitive decisions.

Introduction

In recent years, there has been much interest in the role of metacognition in mathematics education (e.g. Hacker, 1998). Research in metacognition has focused on students' metacognitive awareness during mathematics problem solving. For example, Cardella-Elawar (1992) reports that students trained in learning to monitor and control their own cognitive processes for solving mathematics problems do better than untrained students.

In 1997, the Singapore Masterplan for Information Technology in Education was announced. Its aim is to create an IT-based teaching and learning environment in every Singapore school. The target of the Masterplan is for students to have hands-on use of computers for 30% of their curriculum time. Thus, it appears critical that research should not only explore the development of appropriate software to be used, but also explore the role of teachers and effective pedagogy that can maximise students' learning in IT environments.

This paper reports on one strand of a larger investigation of the effects of metacognitive training on 186 twelve-year-old Singapore students in mathematical word problem solving in a computer learning environment. Specifically, the metacognitive training focuses on activating students' metacognitive processes when solving word problems in the WordMath environment. The primary aim of the investigation is to identify the differential effects of training on these students, and the differences between the students' problem solving processes.

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[1] WordMath was developed by Looi et al (1997). It is a powerful computer learning environment modelled according to the instructional approach of cognitive apprenticeship. It is designed to teach word problem solving to nine to twelve-year-old students in Singapore primary schools using the "model" approach.

Methodology

Five mixed-ability classes of twelve-year-olds from two Singapore primary schools were involved in this intensive study over a period of eight weeks. The classes were randomly assigned to three treatments: one class from each school had explicit metacognitive training before solving word problems with WordMath (MAC); one class from each school solved word problems with WordMath (AC); and one class from one school solved word problems in the traditional "model" method using paper and pencil (TC). For the experimental design, students from all the classes took a written pre-test consisting of ten word problems on the topics Number and Fraction. A written post-test was administered after the students had undergone the two-weeks treatment. There were altogether four training sessions and each session was delivered by the researcher and consisted of a set of learning instructions and word problem tasks. The final written delayed post-test was administered a month after the post-test.

In addition, a pair of students was selected from each class for the case study design. These students had an additional two training sessions in which the students solved four word problems for each session. Their problem solving behaviours were video-recorded but not analysed. The purpose of these additional sessions was to enable the students to feel comfortable working collaboratively in front of the video camera. After these sessions, the pairs of students' problem solving 'think-aloud' protocols of eight word problems on the topics Number and Fraction were video-recorded and these data were later analysed.

Analysis and Results

Quantitative Analysis

Scores obtained from the students' written word problem solving tests were analysed. Table 1 below shows the summary data of their means and standard deviations.

Treatment/ Achievement Scores		MAC (n = 75)	AC (n = 74)	TC (n = 37)
Pre-test	Mean	4.31	2.82	1.68
	Standard Deviation	3.24	2.27	2.06
Post-test	Mean	5.09	3.36	2.24
	Standard Deviation	3.13	2.25	2.35
Delayed Post-test	Mean	5.85	4.34	3.00
	Standard Deviation	3.08	2.30	2.68

Table 1: Means and Standard Deviation of MAC, AC and TC Students' Mathematical Achievements

A repeated measures ANOVA was generated to analyse the data and the results indicated that there is a significant main effect for the treatment ($F = 361.13$, $p < 0.0001$) and there is a significant interaction between the treatment and the mathematics achievement scores ($F = 17.236$, $p < 0.0001$). In addition, as reported in Table 1, the MAC class significantly outperformed the AC class, which, in turn, outperformed the TC class.

Qualitative Analysis

A further analysis was undertaken to try to identify the differences the types of problem solving behaviours shown by the selected pairs of students from the MAC, AC and TC classes while solving mathematical word problems. The following shows the analysis procedures applied to one of the eight problem solving protocols for each pair of students.

Analysis of the MARBLE problem has been chosen because it best illustrates the students' unique problem solving behaviours and collaborative style. The MARBLE problem context is as follows.

Joe Ee, Mun Fai and Jing Hao shared 400 marbles amongst themselves. Joe Ee received 28 marbles. Jing Hao received seven times the total number of marbles Joe Ee and Mun Fai received. How many more marbles did Jing Hao receive than Mun Fai?

Pair Protocols

Schoenfeld's (1985) episode analysis was used to analyse the students' think aloud protocols. Schoenfeld's scheme aims to highlight major strategic decisions made by the students. The think aloud protocol is parsed into episodes, representing periods of time during which the students are engaged in unique types of problem solving behaviour. The behaviours, described in Schoenfeld (1985, p. 297-301), are: reading; analysis; exploration; planning; implementation; and verification. Figures 1, 2 and 3 demonstrate the overall structure of the solution analysis for MARBLE by students HM and XY (MAC), K and SJ (AC), and D and R (TC) respectively.

Key to Symbols

E1, E2, etc = Episode 1, Episode 2, etc; T1, T2, etc = Transition 1, Transition 2, etc

The numbers indicate 'moves' in the protocols (Schoenfeld, 1985).

● Indicates New Information/New Procedure (NI/NP)

▼ Indicates Local Assessment (LA)

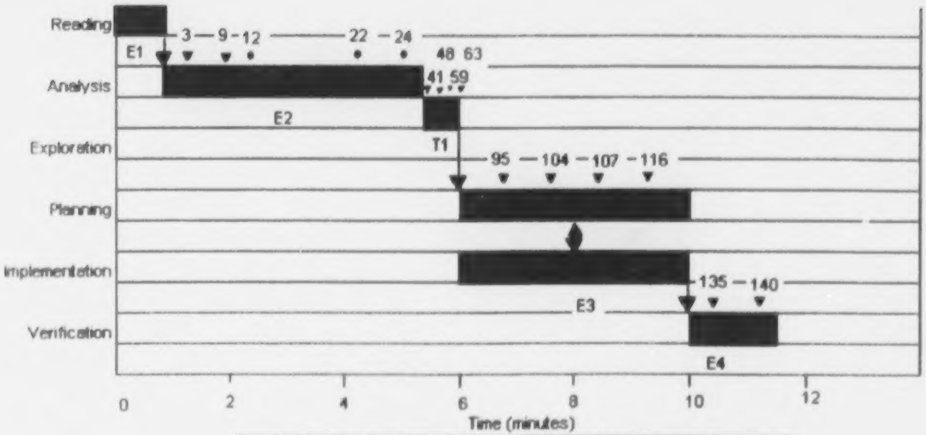


Figure 1: A timeline representation of HM and XY (MAC) solving MARBLE

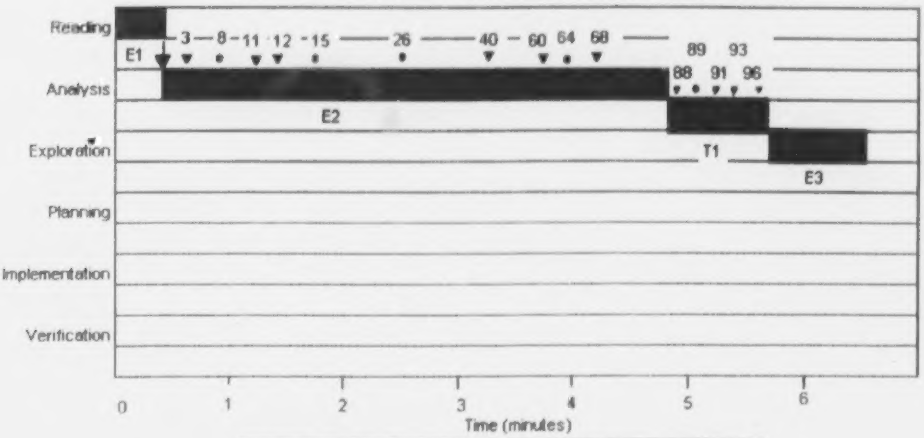


Figure 2: A timeline representation of K and SJ.(AC) solving MARBLE

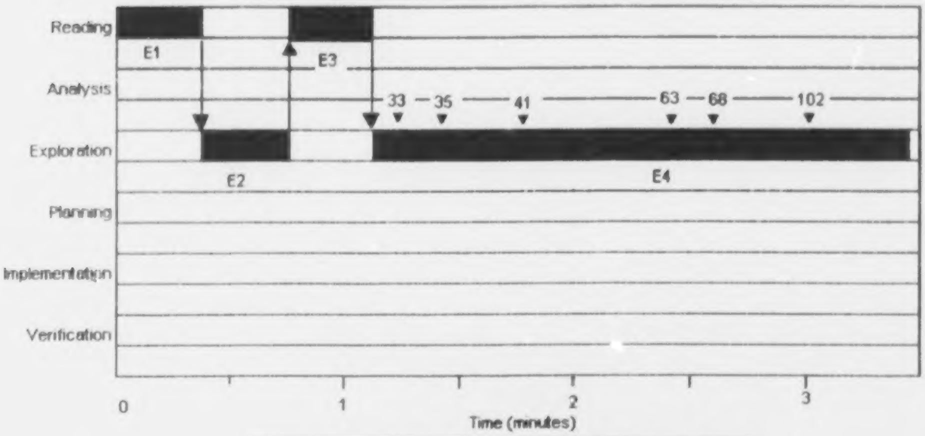


Figure 3: A timeline representation of R and D (TC) solving MARBLE

Summary of Episode Structure

Each figure in Figures 4, 5 and 6 below represents a summary of the characteristic structures of the collaborating pairs of students from the MAC, AC and TC classes for the MARBLE protocol.

Figure 4: Characteristic Structure of HM and XY's (MAC) Problem Solving Protocol

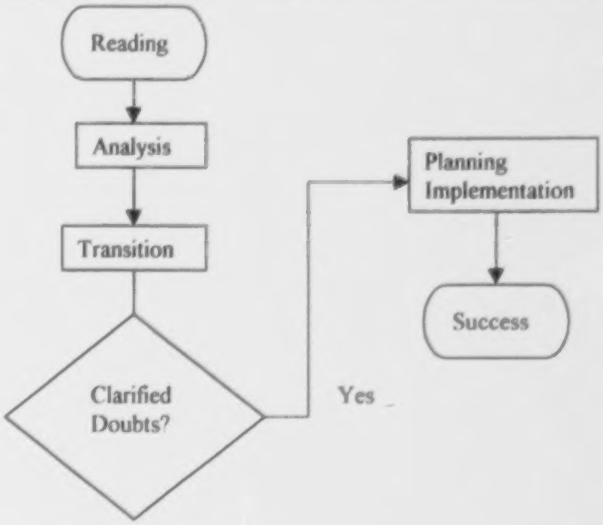

Reading
Analysis
Transition
Clarified Doubts?
Yes
Planning Implementation
Success

Figure 5: Characteristic Structure of K and SJ's (AC) Problem Solving Protocol

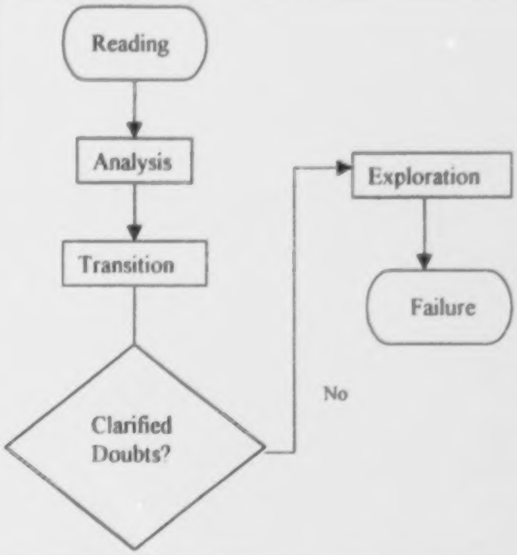
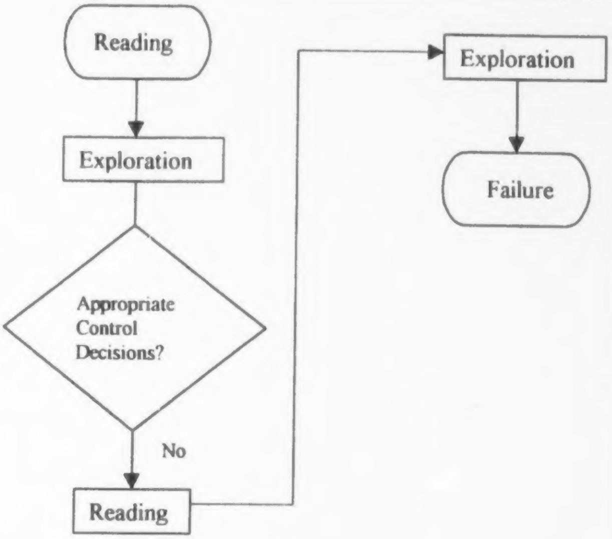

Reading
Analysis
Transition
Clarified Doubts?
No
Exploration
Failure

Figure 6: Characteristic Structure of D and R's (TC) Problem Solving Protocol



The protocol for HM and XY (MAC) could be summarised as a well-regulated progression of activity (Reading → Analysis → Planning ↔ Implementation → Verification) which led to their success in solving the word problem. They also seemed in control of their cognitive actions throughout their problem solving process, as illustrated by the following exchange after the pair had drawn the diagram which represented the word problem.

HM: *The question asked how many more marbles did Jing Hao receive than Mun Fai.*

XY: *So we have to find Mun Fai*

HM: *Let me see (pauses for 3 seconds). This is the unknown {pointing to the diagram} / unknown because of Mun Fai.*
So let say this is one small unit /
XY: *Okay*

The protocol for K and SJ (AC) could be summarised as

1. Reading of the word problem.
2. Accurate analysis of the word problem. This was indicated by the diagram drawn which correctly represented the relationship between the given and the unknown.
3. Transition, in which they failed to clarify the uncertainties they encountered with regards to the goal of the problem.
4. Exploration.

They did not solve the word problem successfully.

The protocol of D and R (TC) could be summarised as

1. Reading the word problem.
2. Exploration, where the pair was observed to take the numbers out of the word problem context and used different operations to manipulate these numbers.

It seemed that they hoped exploration would allow them to obtain a solution. R often engaged D in local assessments, but they did not make appropriate metacognitive decisions which could direct them to the goal of the problem. They also did not clarify each other's uncertainties at all stages of their word problem solving. These weaknesses contributed to their failure in solving the word problem.

Discussion

Empirical evidence from this study re-affirms the value of metacognitive training on mathematical performance in a computer environment. Table 1 shows that there is a significant difference between the performances of the MAC, AC and TC classes on mathematical word problem solving and in a computer learning environment. This indicates that there was a significant differential treatment effect in mathematical performance between students who were exposed to metacognitive training in a computer environment and those students who were taught mathematical word problem solving in the conventional approach. The results from this paper concur with the findings of previous studies that focused primarily on the effects of metacognitive training on mathematical performance in a non-computer environment (Cardella-Elawar, 1992; Clements, 1990; Schoenfeld, 1985).

The analysis of transcripts of collaborative word problem solving of three pairs of students, from MAC, AC and TC classes, using Schoenfeld's episode-based scheme also revealed that students who were made aware of their cognition during word problem solving were in greater control of their problem solving behaviours throughout the word problem solving and their problem solving activities were well-regulated. This led to their success in word problem solving. The poor performance of the AC and TC students may have stemmed from the limited metacognitive techniques employed in solving word problems.

During the study it was observed that AC students generated metacognitive decisions to guide them in their cognitive actions but these decisions were often not appropriate and as a result they were not successful. Yeap (1998) notes that though having metacognitive experiences is deemed to be important, the occurrence of them on their own does little to ensure success in problem solving. He observes that when students set cognitive goals which are guided by metacognitive experiences and metacognitive knowledge, these actions tend to lead to success in problem solving. It was also observed in this study that the activities the TC students engaged in during their solution process had no particular system to follow. They started computation immediately after they read the problem. Schoenfeld (1987) also reported that for these unsuccessful problem solvers with limited metacognitive awareness, mathematical problem solving is often seen as a system of taking one step at a time without any real understanding of the general principle of the problem. Based on the observations of AC and TC students' problem solving process, immediate questions are raised with regard to the types of metacognitive decisions and the role they play in students' mathematical problem solving activities. This issue needs further attention.

Conclusion

This paper supports an approach to instruction which includes metacognitive training in mathematical problem solving in a computer learning environment. As students become aware of their cognitive processes during problem solving, they seem more likely to be able to monitor and regulate their own thinking, which appears to contribute to their success in solving problems.

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USING ALGEBRAIC PROCESSES TO PROMOTE CONCEPT DEVELOPMENT

Anne R. Teppo
Montana State University

This paper presents a vignette of an elementary preservice teachers' classroom discussion focused on finding patterns among lists of numbers having exactly two, three, four, or five divisors. Analysis of the vignette investigates how the algebraic processes of generalizing and abstracting are used to develop a multiplicative structure for whole numbers, and how different forms of representation are used to focus on pattern identification and articulation.

To meet the needs of current reform recommendations for school mathematics, Kaput (1999) characterizes a "new algebra" consisting of five interrelated forms of reasoning, one of which is "algebra as the generalization and formalization of patterns and constraints." (p. 136)

> Generalization involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and the relations across and among them (which in turn become new, higher-level objects of reasoning or communication)."
> (Kaput 1999, p. 136)

This paper analyses a classroom discussion to investigate how a particular activity utilizes such algebraic processes to promote concept development. The discussion, presented as a classroom vignette, also illustrates the theoretical constructs of "didactic object," (Thompson, 1998) "reflective discourse," (Cobb, Bouffi, McClain, & Whitenack 1997) and the notion of abstraction as a process (Mason, 1989). The classroom activity is from a one-semester course for preservice elementary teachers designed to develop mathematical content as well as provide Standards-based learning experiences. Although many of the preservice teachers in the course have had three or more years of high school mathematics, their K-12 experiences have left them with perceptions of a procedurally oriented subject that can be mastered by memorizing the appropriate collections of rules and formulas.

Classroom Activity

The discussion analyzed in this paper was part of an activity aimed at introducing ideas of number theory through an investigation of patterns in natural numbers having exactly two, three, four, or five divisors. During the activity, the preservice teachers worked first individually, then in small groups, and, finally, together in a whole class discussion. The instructor introduced the mathematical content to be investigated, coordinated the small group work, and focused the final discussion.

At the beginning of the activity, the procedures for identifying divisors of given numbers and the terms "factors," "divisors," and "divides" were introduced by asking the preservice teachers to find the four different divisors for the number 15 and the six divisors for 45. In groups of four, they then found different numbers that had exactly two divisors, three divisors, four divisors, or five divisors, and identified patterns to help them add numbers to their lists.

The instructor drew an empty table on the chalkboard with headings for two, three, four, and five divisors. Preservice teachers from different groups were selected to fill in particular columns in this table with the numbers and lists of factors that their groups had found. This shared set of results formed the basis for a whole-class discussion that moved the lesson's focus from calculations with particular number facts to investigations of patterns and structure through the processes of generalizing and abstracting.

During group work, the preservice students did not have difficulty identifying numbers with exactly two, three, or four divisors, but many were unable to find any numbers other than 16 and 81 that had five divisors. Table 1 is an example of the type of information provided by the groups at this stage in the activity. Other numbers were added to the table during the class discussion as the preservice teachers identified and built on patterns.

Table 1. Divisor Table showing the groups' initial entries

2 Divisors			3 Divisors		
#	Divisors	Rule	#	Divisors	Rule
2	1, 2		4	1, 2, 4	
3	1, 3		9	1, 3, 9	
5	1, 5		25	1, 5, 25	
7	1, 7				
11	1, 11				

4 Divisors			5 Divisors		
#	Divisors	Rule	#	Divisors	Rule
6	1, 2, 3, 6		16	1, 2, 4, 8, 16	
8	1, 2, 4, 8		81	1, 3, 9, 27, 81	
10	1, 2, 5, 10				
15	1, 3, 5, 15				

Classroom Vignette

Field notes for the study consisted of a detailed summary of the classroom discussion that was written out by the instructor directly after the end of the class period. This summary paraphrased the preservice teachers' and instructor's utterances and recorded the instructor's actions at the chalkboard. The following vignette is a reconstruction of that activity. While not a verbatim transcript, the vignette indicates the types of statements that were made and the mathematical progression of the discussion. The names (and sex) that have been assigned to each preservice teacher are completely arbitrary since the names of the original speakers were not recorded. This vignette serves as appropriate data for the type of analysis presented in this paper, where the goal of the research was to investigate how the activity promoted the processes of pattern identification and generalization.

The vignette begins after the preservice teachers have written their results on the chalkboard. (See Table 1. "I" is the instructor.)

1. I: Does anyone see any patterns in the lists of numbers on the board?
2. Mary: The numbers with five divisors are squares of numbers with three divisors.
3. I: (*Writes this rule on the board as it was verbally stated.*) Check this rule out with the numbers 16 and 81. Does it work?
4. Several Preservice Teachers: Yes.
5. I: If we use the rule, can we predict another number for this column?
6. John: The number 625 is 25 squared and 25 has three divisors.
7. I: (*Writes the five different divisors for 625 in the 5-divisor column.*) Are there any other patterns in the table?
8. Clarice: All the numbers with two divisors fit the pattern of one and the number itself.
9. Tina: For three divisors, the pattern is one, a prime number, and its square.
10. I: (*Writes these rules out in English on the board.*) Can anyone find a pattern for numbers with exactly four divisors?
11. I: (*There is no response to this question.*) Tina, can you tell us what you meant by "prime number"?
12. Tina: It's any number that has exactly two divisors.
13. I: You can see the pattern in the two-divisor column. The only factors of prime numbers are one and the number itself.
14. I: Sometimes it is easier to see the form of a pattern if the pattern is written symbolically instead of in words. Clarice, can you give me your rule for numbers with two divisors using symbols?
15. Clarice: It's one times x.
16. I: (*Writes on the board* "$1 \cdot x$".) What does x represent?
17. Clarice: Any number.
18. I: Will the number 2/5 work in your rule?

19. Clarice: It has to be any whole number.

20. I: Tina told us that the numbers with exactly two divisors are prime numbers. Let's use the letter p to represent the number in your symbolic rule to remind us that we are using prime numbers. (*Writes on the board* "$1 \cdot p$ *where* p = *any prime number.*")

21. *As the discussion continued, the students established that the symbolic representation for any number having exactly two divisors could be written as* p, *for three divisors as* p^2, *and for five divisors as* $(p^2)^2 = p^4$.

22. I: Can we identify a pattern for numbers with four divisors?

23. Janelle: The pattern goes p, p^2, then we skip a column and it's p^4. The exponent is one less than the number of divisors. The pattern for four divisors is p^3.

24. I: (*Writes* p^3 *above the 4-divisor column.*) Look at the numbers and factors we've listed with four divisors. Let's test Janelle's conjecture.

25. David: 8 is 2 cubed. That fits.

26. I: How about the numbers 6, 10, and 15. They don't fit the pattern. Was 8 just a fluke?

27. Sandra: The number 27 works. Its divisors are 1, 3, 9, and 27.

28. I: (*Writes* 2^3 *next to 8, and adds* 3^3 *and 27 and its divisors to the 4-divisor column.*) Do any other numbers fit this pattern?

29. Alicia: 64 is 4 cubed. Its factors are 1, 4, 16, and 64.

30. I: (*Writes the number 64 and its four factors in the column.*) Let's check Alicia's conjecture. Are there any other factors for 64?

31. Marsha: 64 also has factors of 2, 8, and 32.

32. I: (*Writes on the chalk board the seven factors of 64: 1, 2, 4, 8, 16, 32, 64.*) 64 has 7 factors. Alicia, does this mean that Janelle's rule doesn't always work? Are there some cubes of prime numbers that have more than four divisors?

33. Alicia: I see what it is, 4 isn't a prime number.

34. Brenda: 64 is the cube of a perfect square.

35. I: (*Writes* $4^3 = (2^2)^3 = 2^6$.) If you write 64 in this form it seems to fit the pattern that Janelle found earlier that the exponent is one less than the number of factors.

36. I: Let's go back and look at the other numbers in the four-divisor list that aren't perfect cubes. Do they fit a different pattern?

37. Kisha: The numbers can be found by multiplying two of the four factors together. For example, 6 is 2x3 and 15 is 3x5.

38. I: (*Circles these pairs of factors in the four-divisor column.*) Do these circled numbers exhibit a pattern?

39. Kisha: I was just looking at the list of numbers in the two-divisor column.

40. I: Notice that the circled factors are always both prime numbers. I'd like to conjecture that the non-cubed numbers in the four-factor column can be found by taking the product of two prime numbers. (*Writes the pattern "pq where q is any other prime number."*) You should be able to use this rule to find other numbers to put in the four-divisor column.

At this point, the 50-minute class period was almost over and further discussion was postponed until the next class period.

Didactic Object and Reflective Discourse

The vignette was analyzed to identify instances where students and/or the instructor identified patterns. Note was made of whether the patterns related to particular numerical examples and operations or were generalizations extending beyond these cases. In addition, attention was paid to how the table, written sentences, and algebraic symbols were used to direct the pattern search.

The activity's focus on finding patterns was facilitated by the choices and sequencing of different forms of representation – the divisor table, written records of natural language, and algebraic symbols. At the beginning of the discussion, the divisor table served as a "didactic object" that encouraged students to employ mathematical processes and introduced the concepts of interest. Thompson (1998) defines a "didactic object" as something to talk about that is designed to support reflective mathematical discourse. He points out that objects are not didactic in and of themselves but are so "in the hands of someone having in mind a set of images, issues, meanings, or connections affiliated with it that need to be discussed explicitly" (p. 11), and because "of the conversations that are enabled by their presence" (p. 8)

The instructor's request to identify patterns in the lists of numbers in the table focused the class discussion on a multiplicative decomposition of numbers. For example, in lines 2 and 6 the students described particular cases where numbers were composed by squaring. The notion of a prime factor was discussed in connection with numbers having exactly two or three factors in lines 8 through 13. The multiplicative relationship of the prime factor to the given number became explicit as Clarice (line15) re-expressed her rule from line 8 as, "It's one times x."

Symbolic language made it possible to stress this idea of a multiplicative relationship while at the same time ignoring particular operations such as squaring. Algebraic symbols drew the students' attention to a new set of objects - patterns of operations expressed in exponential form. The discussion moved from the activity of describing individual rules to thinking about the pattern common to all the rules when, in line 23, Janelle expressed this pattern as, "The exponent is one less than the number of divisors." From that point forward, the instructor was able to keep bringing the discussion back to this level of thinking. In lines 24 through 35 there is a "zig-zag between the general and the particular." (Cobb et al 1997, p. 273) Janelle's general rule is tested as a specific column rule and the results are then used to extend and exemplify the general rule for the case of seven divisors.

The discussion in the vignette illustrates the notion of "reflective discourse", defined by Cobb et al (1997) as being "characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion." (p. 258) Two such shifts occurred in the activity. At the beginning of the vignette, the preservice teachers described the patterns they had used, at an implicit level, to construct their divisor lists. Later, the discussion moved from a focus on the column rules to identifying the overall pattern across individual columns that had been implicit in their identification of the earlier rules.

In line 12, the instructor implied this second shift when she said it was easier to see the "form of a pattern." However, her request to Clarice to give a rule for numbers with two divisors in symbolic form kept the discussion focused on describing the procedures for each individual rule. The difference in the subsequent discussion was that by using symbols to express the individual column rules, instead of the previously written natural language descriptions, the instructor was making it possible for the students to also make a second shift in thinking.

Abstractive Processes and Concept Development

The processes of making and articulating generalizations that were used in the vignette are an integral part of reflective discourse and illustrate Mason's (1989, as described in English & Sharry 1996) steps in the development of abstraction. In his sequence, students experience operations, express this experience, articulate the properties of the experience as generalities, and finally, manipulate the expressions to identify new properties.

The preservice teachers *experienced* operations in the pre-vignette activity as they used multiplication and division facts to create individual lists of divisors. This experience was then *expressed* via organized lists in the divisor table and articulated as particular examples, for instance, in lines 4 and 6. The *properties* of this operational experience became the focus of the class discussion and were *articulated* through natural language descriptions of individual column rules, as in lines 2, 8, and 9.

Mason's final step - here the abstraction of and use of the notion of a composite number was not reached during the activity. However, progress in the development of a concept image for composite number occurred as the preservice teachers utilized, identified, and articulated the properties of this concept, as, for example, in lines 13, 23, and 37.

The pattern-seeking activity also utilized Kaput's (1990) description, given earlier, of algebraic processes. Through these processes, the discussion moved from a procedural towards a structural orientation. (Sfard & Linchevski 1994) The preservice teachers began the activity at a procedural level as they worked with particular numerical computations. The use of algebraic symbols moved the focus of attention to a structural level as the preservice teachers noted patterns shared by sets of similar operations. At this level, the notion of a functional relationship between composite numbers and their respective number of divisors was implicit in Janelle's generalization in line 23 that "the exponent is one less than the number of divisors."

Conclusion

Mathematics has been described as "both an *object* of understanding and a *means* of understanding." (Romberg & Kaput 1999, p. 6) The vignette illustrates the interconnected nature of this characterization whereby the *processes* of generalizing and formalizing patterns were the tools through which *properties* of composite numbers became the explicit objects of thought. In the classroom activity, algebra was not the object of study, but a crucial pedagogical component for concept development.

Even though the instances of reflective discourse were very short, the activity was carefully structured to maximize the potential for utilizing algebraic processes by the instructor's progression from written sentences to symbolic notation. The vignette illustrates how this structure enabled the preservice teachers to become active participants in the processes of generalizing and abstracting and provided experiences necessary for a later encapsulation of the notion of composite number.

Zaskis and Campbell (1996) have noted that elementary preservice teachers exhibit a highly procedural orientation to concepts of number theory. They found that individual preservice teachers tend to think "of prime decomposition as a factoring process, rather than conceptually in terms of a number expressed as a product of primes." (p. 211) The classroom activity described in this paper provides a vehicle for making the transition to a more conceptual understanding through active reflection on the process and properties associated with number theory.

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CHILDREN'S LEARNING OF INDEPENDENCE: CAN RESEARCH HELP?

John M. Truran
Private Practice, Adelaide, South Australia

A Handbook Model is used to codify research results about children's understanding of independence so that it is accessible to practising teachers and indicates useful future research. Given the imprecise world of the classroom, such an approach requires the development of appropriate criteria for linking research and practice .

The well-known and wide-spread "Gambler's Fallacy" is a misconception about statistical independence which uses a "Negative Recency Heuristic" to posit, for example, that the greater a run of heads from a fair coin, the greater the probability of tossing a tail on the next throw. What does research have to say about how children develop efficient constructions of independence, and how might this provide psychological insights about learning mathematics for a practising teacher?

The Handbook Model

One answer to this question may be found by using the "Handbook Model" (J. Truran, 1998; J. & K. Truran, 1999), which uses conciseness, brevity, and reliability to achieve two parallel purposes.

The first makes the findings of researchers available to practising teachers in a readily accessible form. Research suggests that most classroom teachers have very little exposure to new findings and tend to resist those they do meet (e.g., Haimes & Malone, 1993; Swinson, 1993). The Handbook Model provides a level of authority for classroom practice which is wider than any individual teacher's experience.

The second contributes to Kuhn's "model building" stage of "normal science" which Romberg (1983) sees as appropriate for educational research as well as the hard sciences, even though educational science is much less developed. This model posits that, after a long period of fact collection, confrontation, confusion and reflection, a small group of scholars may construct a particularly clear synthesis which isolates the key variables, attracts adherents, and raises open-ended questions for further work. Synthesis in stochastics education research is particularly difficult because it is done within different disciplines different epistemological theories, thus hindering communication between workers (Shaughnessy, 1992, p. 467).

But educational research is more than pure science: it is frequently done (often by practising teachers) to enhance classroom practice, and must satisfy not only the demands of pure science but also the needs of classroom teachers. Educational researchers need to plot a difficult course between the Scylla of academic accuracy and the Charybdis of pedagogic pragmatism. This requires compromise, but the current imprecision of much classroom practice means that the level of rigour required of pure results before they may be usefully be applied is less than in some other applied disciplines. Academic rigour must be neither denigrated nor disregarded, but it must be seen in a broader context when pedagogy is an object of academic study in its own right. As Balacheff (1997) has observed:

> [p]sychology is only part of the relevant approaches to the problems raised by mathematics learning and teaching, and for example one must be able to take teaching processes as an object of study as such, as well as the epistemology of mathematics from a teaching/learning perspective.

This paper aims to codify research findings about the learning of independence in a way which can assist stochastics education research to move closer to Romberg's third stage of normal science and also be of practical use to classroom teachers.

The Instability of Many Stochastics Research Results

In stochastics research allowances must be made for inconsistent results when small changes are made to questions. E.g., Konold et al. (1993) presented four sequences of outcomes from five tosses of a coin, and asked which was *most* likely or whether all were equally likely, and then asked which was *least* likely or whether all were equally likely. While 72% of subjects correctly answered the first question, only 38% of them did so for the second. Again, when comparing probabilities children often use different, unpredictable heuristics influenced by the size, familiarity or simplicity of the numbers involved (J. & K. Truran, 1999), or give different responses to written questionnaires from what they say in discussions (Alarcon, 1982).

Unstable responses may also arise because children may not see the equivalence of mathematically identical random generators (RGs) with different structures such as a standard die and an urn containing six identical balls numbered from "1" to "6" (K. Truran, 1994), or standard dice of different sizes (Wilkinson & Nelson, 1966). Inconsistent responses may be more prevalent with markedly asymmetric RGs (J. Truran, 1994) or when their culture attributes some chance events to luck and others to God (Amir et al., 1999, p. 30).

Classical and Trial Independence

In much statistics teaching the single term "independence" is used to describe two quite different concepts (J. & K. Truran, 1997). One is "Classical Independence" which describes events which are subsets of the possibility space of a specific RG. Two events A and B are said to be independent if and only if $pr(A \cap B) = pr(A) \times pr(B)$, and classical independence does not imply any causal relationship whatever, although it is often suggested that "A has no effect on B" is an equivalent definition of independence. But this confuses "Classical Independence" and a quite different meaning of independence which is here called "Trial Independence".

"Trial Independence" occurs when the operation of an RG has no effect on the *randomness* of the operation of another RG or of a subsequent operation of the same RG. This cannot be verified mathematically, but is subjectively decided using past experience or physical features of the RGs. Thus Freudenthal (1973, p. 604) saw two dice joined by string of varying lengths as having varying degrees of influence over each other's outcomes. Trial independence requires a subjective decision about randomness, which is exactly how the Gambler's Fallacy is derived. It is the quality of the supporting data which determines the quality of the decision. Because the subjective nature of trial independence has been under-emphasised in statistics, the need for supporting data has not been well examined by researchers.

Classical independence in its pure form has been little studied, although some researchers have addressed the related, more complex concept of statistical association (e.g., Estepa et al., 1994). So this paper will discuss trial independence only.

Research Summary

There is no space here for a comprehensive summary of the three main approaches used, but a representative selection has been chosen to form a basis for the later section discussing how research and pedagogy may be validly linked

Analysis of Runs

A common approach has presented subjects with several equally likely sequences of outcomes from an RG and asked which is most likely or whether all are equally likely. Students are believed to consider sequences with no obvious pattern as being more likely, following Kahneman & Tversky's (1972) interpretation that such sequences are seen as less "representative" of the total set of possible outcomes.

But Konold et al. (1993) have used their divergent results mentioned above to argue that subjects interpret different forms of questions in different ways. Thus subjects interpret their "most likely" question as requiring a prediction of which outcome will happen—an "Outcomes Approach"—but use a representativeness heuristic for their "least likely" form. An Outcomes Approach is seen as needing less cognitive processing than a more holistic approach, and therefore likely to be the default heuristic adopted. This matches Goodnow's (1958) view that people's concern is more with correct prediction than with understanding structure. But Konold et al. have not shown *why* their subjects see these two very similar situations as different.

With similar questions and younger students Batanero et al. (1994) found similar results, but smaller discrepancies between question forms. They suspected that Outcomes Approaches were used, but saw the lack of consistency between questions of similar type but different context as important evidence that few students used "normative reasoning" of any sort. Furthermore, Amir et al. (1999) showed that children's subjective probabilities of, for example, five "H" and one "T" (0·420) are not so much larger than those for "H-H-H-T-H-H" (0·357) that there is convincing evidence that children clearly discriminate sequences from classes of sequences.

Prediction of Outcome

Another approach has asked for predictions of the next outcome from an RG. Asking for a prediction is logically meaningless, but different responses to such questions in different situations may still shed light on subjects' understanding of chance.

Thus Fischbein (1975, ch. 4) discerned "Probability Matching Behaviour" in young subjects' predictions for an RG whose structure was unknown to them, and posited the existence of an intuition of probability. The logical difficulty may be avoided by asking subjects which outcome they would choose if they *had* to make a choice. All such predictions are subjective responses, and thus related to the Gambler's Fallacy.

Fischbein (1975, p. 59) argued that children predict random events by using the "Negative Recency Heuristic" of the Gambler's Fallacy, but others have shown that they may sometimes employ a "Positive Recency Heuristic" (Green, 1983) or a Negative Recency Heuristic for predictions, rather than outcomes, and some may employ a Pascalian response of consistently predicting the most likely outcome, regardless of the actual outcomes (J. Truran, 1996). The circumstances under which the different heuristics are used have not been clarified, but the use of Pascalian responses by naïve children from age 9 upwards does raise questions about whether the variety of responses obtained is a consequence of the experimental form used.

Ayres & Way (1999) have developed a more controlled experimental form using a video of deliberately constructed "random" draws. This has been well received by children and revealed an increase in Negative Recency Heuristics when there are many more refutations than theory would predict, but the methodology uses predictions only at every fifth draw, so is not easily compared with other results.

Simultaneous Operation

A third approach developed by Fischbein et al. (1991) asked whether a certain pair of outcomes from two RGs would be more likely when the RGs were operated concurrently or consecutively. Other similar research has found a variety of responses, but confirmed that many incorrect responses choose ways seen as providing the greatest "control" over outcomes (K. & J. Truran, 1999). Children's beliefs that single RGs may be "controlled" have often been reported (e.g., Wollring, 1994); their use in these more complex situations suggests that letting go of "control" may be fundamental for developing the received understanding of trial independence.

Implications for Pedagogy

We now have a sufficient basis to show how classical research may be interpreted for classroom practice. Thus all the "Analysis of Runs" results confirm that many students poorly discriminate sequences from classes of sequences and/or are influenced by the form of a question form. So these may be taken as well established facts about common misconceptions which teachers might expect to encounter. Although the research has not established well-proven reasons for the marked influence of question form, it has shown that different contexts and the process of explicitly examining sequences and classes of sequences concurrently do provide further insights into how children think about runs and independence. While they are not supported by experimental evidence about pedagogy, they are sufficiently rigorously based to suggest that changing pedagogical approach to incorporate these findings may encourage good learning.

These results exemplify the pedagogical theory on which the Handbook Model rests. For example, Skemp (1971/1986, ch. 2) emphasised that many embodiments of a new concept are needed before it may be securely learned. For Analysis of Runs, research has shown that different approaches yield markedly different responses. This strongly suggests that such approaches might form a useful part of classroom practice. Pure research tempered by education theory can provide a responsible basis for improved practice which balances cautious pragmatism and realistic leaps of faith. This will be the basis of the Handbook Model which concludes this paper.

Again, pragmatism suggests other useful insights. Because for Prediction of Runs many incorrect heuristics are employed, apparently indiscriminately, and some naïve children do construct a correct heuristic, it seems wiser to use activities which encourage appropriate heuristics rather than false ones. Similarly, because "control" is a marked feature of the Simultaneous Operations findings, it seems wise to encourage situations where it is difficult to plead control, bearing in mind that the idiosyncratic use of use of conflicting approaches means that approaches based on cognitive conflict approaches are unlikely to be successful (Konold et al., 1993).

A few researchers have tested potentially valuable approaches with children. Green (1987) developed a computer-supported set of data, some from biased RGs, to encourage students to analyse systematically some aspects of outcomes from an RG, such as the length of runs. This clarified children's ideas about both randomness and bias. Burgess (1999) has obtained evidence that race-track games may persuade students to change their probabilistic beliefs, partly because they require several throws before enough data has been obtained to arouse cognitive conflict. But there has been little research into pedagogical practices like these.

One problem not addressed here is the complex relationship between randomness and trial independence. Here trial independence has been defined as a situation having no effect on randomness. But Green saw his work as teaching about randomness and others (e.g., Pratt & Noss, 1998) have seen an understanding of trial independence as contributing to an understanding of randomness. Perhaps the two really have a "chicken and egg" relationship; this is a pedagogical issue which there is not space to examine here.

A Handbook Model for Trial Independence

As for previous papers in this sequence, to save space and highlight main themes references are omitted and terms defined in this paper are not re-defined. Some ideas, such as "control" and mathematical equivalence of different RGs, are not discussed here, but would obviously find a place in a complete Handbook.

The Teaching and Learning of Trial Independence

Students are prone to using a variety of approaches in inconsistent ways when presented with independent RGs. Many already have a belief that they can "control" individual operations of an RG and carry this over to a belief that they can "control" the operation of several RGs as well.

The ideas of randomness and trial independence are closely linked, and will not be easily separated in the classroom. The ideas of bias and asymmetry may also be encountered during experiments, and should be carefully distinguished from trial independence.

There is a paucity of research into effective practical ways of teaching trial independence, so this summary describes similar or identical situations which are seen as quite different by many children. They certainly need to experience these different situations and to compare and reflect on them, but research evidence about the most effective approaches is limited.

At least three types of situations illustrate trial independence. Present these with a variety of RGs of varying degrees of symmetry. Provide pure examples and also examples embedded in a real-life context, preferably in a form where parallel cases may be compared. Include examples like gender of new-born children whose interpretation may be subject to a variety of significant cultural influences.

Examination of Sequences of Runs from a Single RG

Some sets of outcomes are seen as more or less representative of all possible sets than others, so students need experience in constructing sequences, and in interpreting sequences constructed by others. Problems which involve analysing sets of long sequences to determine which have been fabricated have been proved to be effective and popular.

Making Predictions about Outcomes from an RG

These situations may bring out the Gambler's Fallacy, but they may bring out many other false heuristics as well. All of these heuristics are subjective and difficult to test by setting up counter-examples. But understanding of trial independence requires the making of subjective judgements, ones based on sound data bases. Race-track games have been found to challenge children's misconceptions, and they encourage children to look at sets of outcomes from an RG, rather than individual ones. Video filming now makes it possible to construct sequences of draws which look random but which have in fact been fabricated.

Considering Pairs of Operations of RGs

Many children do not see concurrent operation of two RGs as being mathematically equivalent to consecutive operation. Although their choices are often based on their perceived ability to control outcomes, they differ in their choice of preferred approach.

These three situations are superficially quite different, so it is necessary to emphasise their common element, viz., the need for a subjective judgement based on good evidence about whether an operation of a specific RG affects the randomness of other concurrent or consecutive operations.

Conclusion

This Handbook Model gives approaches quite different from many classroom texts. This suggests that research can help the teaching of independence, and, because the Model also makes clear some important gaps in the research corpus, it fulfils one of the criteria for model building—the raising of new questions. It is not the quantum leap which Romberg implies is a necessary part of model building, but it effects a partial and applicable synthesis, so does form part of the model building process.

The Handbook Model, of course, is not a total recipe for classroom practice. It does not address issues of the relationship between teacher and student such as might be found in Brousseau's (1970–1990/1997) "didactical contract". Nor does it use a cognitive framework as a model for classroom practice as has been done, for example, by Jones et al. (1996). Rather, it sees research findings as *one* of the forces which should drive pedagogical practice. In practice, they are frequently neglected, yet they provide a link between psychology, mathematics and education, which can be both accessible and valuable both for current practice and also for future research.

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Advancing Arithmetic Thinking Based on Children's Cultural Conceptual Activities: The Pick-Red-Point Game

Wen Huan Tsai
National Hsin-Chu Teacher College, Taiwan
tsai@cc.nhctc.edu.tw

ABSTRACT

This research described how children link outside and in-school mathematics based on cultural conceptual activities through the cultural conceptual learning and teaching model (CCLT). Findings are presented to address four questions: (1) what kind of knowledge do children bring into classroom when they involved in cultural conceptual activities? (2) How does this informal knowledge interact with the formal knowledge in classroom teaching? (3) What are the difficulties encountered by teachers in order to facilitate students' grasp of mathematical meaning? And (4) what effect does the CCLT teaching program have compared to a regular teaching program on children solving problems in school and task problems outside of school?

Introduction

In recent years, many studies have focused on mathematical cognition related to individual competence in daily life context (Carraher, 1988; Lave, 1988; Saxe, 1991; Bishop & Abreu, 1991; Tsai, 1996). A review of children's out-of-school mathematics raises critical questions about how children come to form mathematical understanding and about the interconnection between informal knowledge out of school and formal knowledge in school. (Millory, 1994; Hibert & Carpenter, 1992; Resnick, 1987). According to Resnick's viewpoint (1987), teachers should concern the role of cultural aspects that are meaningful ways for students to make sense of the abstract symbols of school mathematics. She emphasized that culture contributes to better understanding in students' learning and it therefore needs to be integrated into mathematics teaching. This research hypothesized that established the link between children's cultural activities and school mathematics will improve children's mathematics learning in school as well as their ability to solve daily mathematics problems out of school. As a result, four questions will be discussed in this paper: what knowledge children bring into classroom when they involved in cultural conceptual activities; How this informal knowledge interacted with the formal knowledge in classroom teaching; what are the difficulties encountered by teachers in negotiating mathematical meaning with students; and what are the teaching effects between the CCLT program and a regular teaching program on children solving problems in school and task problems outside of school.

The Cultural Conceptual Learning Teaching Model (CCLT)

Central to this study is the view that an understanding of mathematical meaning is to connect different learning environments or situations (Greeno, 1991). Brown, Collins, & Duguid (1987) also emphasized the importance of the relationship among activity, concept and culture and learning must involve all of them. Hiebert & Carpenter (1992) also proposed that children's informal knowledge could serve as a basis for the development of understanding of mathematical symbols and procedures in school setting, regardless of the content domain. Based on these points of view, this study developed a learning teaching model called the Cultural Conceptual Learning Teaching Model (CCLT) (Tsai, 1996) that tries to combine individuals, activities, concepts, and culture together. CCLT contains three learning environments: construction environment, connection environment, and practice environment and six learning stages: playing stage, constructing stage, connecting stage, re-application stage, practicing stage, and reflection stage.

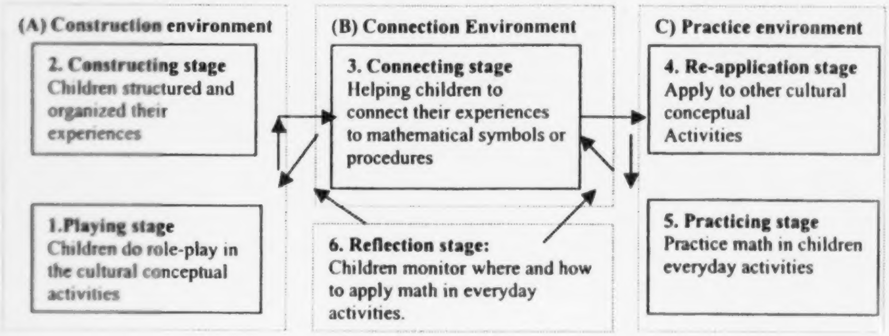
The playing stage provides some cultural-conceptual activities for children to do role-plays. In this stage, children share, negotiate, and construct their immediate experiences to achieve the emergent goals of arithmetic problems with peers and old comers (teacher or expert children). In the constructing stage, the teacher designs a guide sheet that has structural objectives that need to be accomplished by students. In the connecting stage, based on children's experiences or strategies, teacher tries to help children construct a connection of their experiences to mathematical symbols and procedures. In the re-application stage, teacher provides another similar cultural-conceptual activity for children to re-apply the learned mathematical concept. In the practicing stage, children try to practice school mathematics in everyday situations by using opportunites provided for them. In the reflecting stage, children are trained to monitor their thinking and be aware of where and how they can apply school mathematics in everyday activities.

This learning environment contains more than one level of learning. The author calls this learning model as Cultural Conceptual Learning Teaching model (CCLT). In CCLT, there are four kinds of cultural activities integrated into classroom teaching as follows: the Pick-Ten-Point game*, counting lucky money in a red envelope (Chinese traditional hobby in Chinese New Year), shopping and selling toys, and monopoly activity. This paper focuses on the findings of the playing of the Pick-Red-Point game only.

*** The Pick-Red-Point Game (PRP)**

The Pick-Red-Point game is a popular pork game played in Taiwan. Most people play this game in the Eve of Chinese New Year or on holidays of festivals. According to survey, there are 59% of second and third graders who know how to play this game in Hsin-Chu, Taiwan. Forty cards, from 1 (A) to 10 with four suits, will be used. Four players, each of whom is dealt six cards, play the game. Among the rest cards, four cards are faced up and the others are faced down on the table. A player needs to match a card from his hand with another card from the table in order to produce a pair of card that sum up to ten. Otherwise he needs to put one of his cards facing up on the table and then turn a card up from the deck and wait for the next turn. After all cards are paired, each player counts how many red points he or she get. The one who got the highest points is the winner.

Figure 1: The Cultural Conceptual Learning Teaching Model (CCLT) (Tsai, 1996)


(A) Construction environment
2. Constructing stage
Children structured and organized their experiences
1.Playing stage
Children do role-play in the cultural conceptual activities
(B) Connection Environment
3. Connecting stage
Helping children to connect their experiences to mathematical symbols or procedures
6. Reflection stage:
Children monitor where and how to apply math in everyday activities.
C) Practice environment
4. Re-application stage
Apply to other cultural conceptual Activities
5. Practicing stage
Practice math in children everyday activities

Methodology

There are sixteen second-grade classes in Hsin Chu city participated in this study. Eight classes were randomly assigned to the treatment group and the rest to the control group. Teachers from the control group met in a half-day workshop that focused on the arithmetic content of the textbook. On the other hand, teachers from the treatment group met together every Friday to design the cultural activities and also to share their teaching experience. Every instruction was observed and videotaped and students' worksheets and journals were also collected and analyzed.

Two tests, The Standardized Test and Three Test Conditions, were conducted to test the teaching effects of different teaching approaches. The Standardized Test was used as the covariate for the two different groups. The Three Test Conditions Test contains six computational problems (for example, 2+3+7+6+8=?), six word problems (for example, John's house has seven dogs, two cats, three birds, and eight rabbits. How many pets are in John's house totally?), and six task problems (For example, the interviewer provided the numbers of diamonds or hearts with 2, 3, 7, 6, 8, then asked children to count the total points). Four students were randomly chosen from four levels on the Standardized Test as interviewees and their strategies for solving the task problems were recorded and analyzed.

Results

1. What kind of knowledge do children bring into classroom when they play the PRP game?

There are two "emergent mathematical goals" that children bring in playing the PRP game. First, children need to match a card from his hand with another card from the table in order to produce a pair of card that sum up to ten. Four strategies were identified:

some children counted by fingers, some counted by the card points, some counted by mental, and some experienced players were able to count automatically. Second, at the end of game, children needed to count the total points they have gotten. Children did not learn how to add the sum more than one hundred in school so far, but there were four strategies that children used to solve the problem: (a). Counting by composing tens (CT): Children are grouping cards with ten points, and then they count by one ten, two tens, three tens, and so on, and then they continuously count the left. For example, suppose the cards numbers are 6, 7, 6, 9, 5, 4, 3, and 1, then children will group those cards into (6,4), (7,3), (1, 9), 6, and 5, and count one ten, two tens, three tens, thirty-six and forty-one. (b). Counting by fingers (CF): For example, suppose the card numbers are 6,7,9,4,and 3, then Children count by using their fingers and read it out one, two, three… twenty eight, and twenty-nine. (c). Counting by mental (CM): Children count the total points mentally. For example, suppose the card numbers are 6,7,9,4,3, and then children read out six, thirteen, twenty-two, twenty-six, and twenty-nine. Children will stop a little while when they speak out a number. (d). Counting by points (CP): Children count by looking at the dots of the card points and count them out: one, two, three, four…twenty-nine. During play, if some children count too slowly, they will be pushed by other students to make a decision quickly or give up for other's turn.

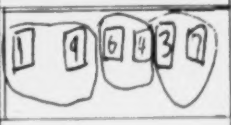
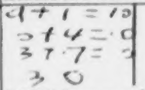
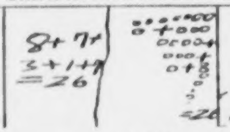
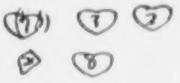
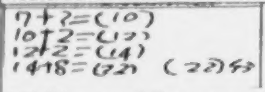
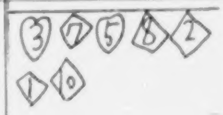
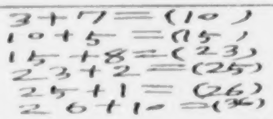
2. **How does children's informal knowledge interacts with formal knowledge in classroom teaching?**

The evidences show that there are some gaps between strategies using to count the total points in PRP and strategies using to calculate the sum on a worksheet. Although there are some students used the methods consistently in theses two situations, many are not. Table 1 shows that S1 used the CT strategy in PRP and used the similar method in worksheet project. S2 also used the consistent strategy in both situations to figure out the sum of card numbers. But S3 and S4 used inconsistent strategy in different situations. S3 used CT strategy to count the total points in PRP, but used CF strategy to count the sum of the card numbers in worksheet project. S4 used CT to count the total points in PRP but used CM to add the sum of card numbers in worksheet project.

As mention above, we have found that there are some gapes between the strategies used in different situations. In CCLT program, it provides many chances for children to connect those different forms of knowledge or strategies. In the construction learning environment, children were treated as a "whole person" to solve problems independently and teacher as a coacher or guider. In fact, children kept their eyes on each other when one counted one's own points. Thus they might change their strategies unawares. Moreover, some children told others how to count the total points quickly.

In the construction and connection environment, teacher provided opportunities for children to share and compare their own approaches or strategies in different situations and share their works for classroom discussions. It let children to find the similar or difference between their approaches gradually.

Table1: Comparing the Strategy used in counting points and calculating the sum of card numbers

Student	Strategy used in PRP	The numbers of cards	Strategy used to calculating the sum of the numbers on worksheet.
S1	CT strategy		 *Children used the similar strategy as CT
S2	CP strategy	8,9,3,1,7	 • Children used "o" to present one point then count how many "o" and wrote down answer is 26.
S3	CT Strategy		 7+?=(10) 10+2=(12) 12+2=(14) 14+8=(22) (22) *Children used finger to count
S4	CT Strategy		 3+7=(10) 10+5=(15) 15+8=(23) 23+2=(25) 25+1=(26) 26+10=(36) *Children count by mental and wrote down the answer.

3. **What are the difficulties encountered by teachers in order to facilitate students' grasp of mathematical meaning?**

Although this study did not focus on individual teaching, some issues were raised during the meetings with teachers of treatment group and classroom observations. In the beginning, one teacher said that this teaching program wasted time and asked why we did not teach students directly through the traditional method. Another teacher complained that students were too loud and not disciplined in the playing stage and the construction stage. But all teachers felt excitement and unbelievable when their students used various strategies that they hadn't see children using before. In all cases, teachers said that their students requested to "play" those activities again.

With cultural activities, knowledge derived from situations in everyday problems. Therefore, students offered "informal" explanations for their solutions on specific problems. Their teacher according to normal mathematical reasoning did not always accept these "reasonable" explanations, but they could not refute it. In this case, teacher's authority was challenged and they needed to negotiate their meaning with children. Teacher and students will then became equally to take and share their knowledge or experiences based on the rich of children's cultural experiences.

4. **What effect does the CCLT teaching program have compared to a regular**

teaching program on children solving problems in school and task problems outside of school?

(1) The Standardized Test

The Standardized Test was administrated before teaching and treated as the covariate for the other test. However, there is no statistically significant difference between the CCLT group (M=72.89; SD=17.36) and the control group (M=72.09; SD= 17.61) on the Standardized Test (F=0.25; P>.05). Therefore, the Standardized Test cannot be used as the covariate for comparing the CCLT group and the control group.

(2). Children's achievement in three test conditions

In this study, we have two hypotheses: (H1) the CCLT group will score higher on the task problems than the control group; (H2) the control group will score higher on the computation problems and word problems than the CCLT group. From Figure 2, the hypothesis H2 has been rejected since the scores of the CCLT group are higher than the scores of the control group on all three tests.

Figure 2: The Plot of Relation Between Treatments and Testing Conditions

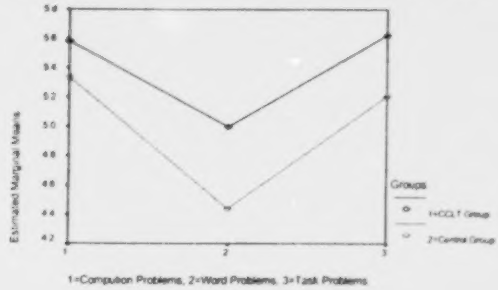


Table 2: Analysis of Variance of Comparison in the Different Test Conditions By Treatment on the Overall Achievement of Testing Conditions

SV	SS	DF	MS	F	P
Between Subjects					
Treatment	12.09	1	12.09	5.20	.025*
Residuals	218.30	94	2.32		
Within Subjects					
Conditions	33.15	2	16.57	30.25	.000***
Treatment × Conditions	1.17	2	.059	1.07	.345
Residuals	103.01	188	.55		

*P <. 05; **p < .01; ***p < .001; Treatment: CCLT Group and Control Group
Conditions: Computational Problems, Word Problems, and Task Problems

Table 2 shows the results of 2 (treatment vs. Control) × 3 (computation problems, word problems, and task problems) repeated measure design. It revealed that the main effect of treatment had significant difference between the CCLT group and the control group. Moreover, there was no interaction between the treatment and the testing conditions. It means that students who learn mathematics based on their cultural activity through the CCLT teaching program not only improve their learning of mathematics in school but also can improve their solving on the pork task problems in their cultural activities.

(3) Children's strategies used in solving the task problems

There are four strategies identified when children solve the task problems: Counting by composing tens (CT), counting by fingers (CF), counting mentally (CM), and counting by points (CP) and others. Table 3 summarizes the means, standard deviation of frequency used by children, and ANOVA analysis between group differences in solving the six task problems. The results showed that children in the CCLT group used the CT strategy to solve the task problems more often than children in the control group. Conversely, children in the control group used CF and CM strategies more often than children in the CCLT group.

Table 3: Summary of Means, Standard Deviation, and ANOVA Results For Between Group Differences on Solving the Task Problems

Strategies	CCLT M (SD)	Control M (SD)	F
CT	3.12(1.78)	1.10(1.51)	35.741***
CF	.771(1.35)	1.58(2.10)	5.063*
CM	.667(1.01)	1.48(2.17)	5.503*
CP	.729(1.67)	1.35(2.13)	2.544
OTHERS	.604(.893)	.457(.874)	.654

*P<.05 ; **P<.01 ; ***P<.001

Conclusion

From the evidence of this study, learning arithmetic through children's cultural activities not only affects children's learning of school mathematics but also improves their ability to solve the task problems in real life. This result reconfirms the effect of the CCLT teaching model in an earlier study (Tsai & Post, 1999). This results supported Hiebert's proposal (1988) that in school settings children's informal knowledge can serve as a basis for the development of understanding of mathematical symbols and procedures, regardless of the content domain. One of possible reasons is that this study chosen the most popular cultural activities for classroom teaching, therefore, children brought lots of experiences to be shared. Another possibility is that the CCLT model provides a learning environment for children to connect their everyday experiences to school mathematics and to practice school mathematics in everyday activities. Thus it helps students to promote the integration of children's informal knowledge and formal knowledge and which is seldom happened for those teaching that based on a more traditional textbook. However, there are some limitations on this CCLT learning and teaching model. One of weakness is that it is not easy to design cultural activities to support children's learning for all mathematical concepts. Another problem is teachers' adoption on this teaching model. Teachers' authority was challenged and they needed to negotiate meaning with children. Teacher and students became equals in socially sharing and taking knowledge based on children's cultural experiences.

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THE INTUITIVE RULE *SAME A-SAME B*: THE CASE OFAREA AND PERIMETER

Pessia Tsamir* ** and Nurit Mandel**
Kibbutzim Teacher College* Tel Aviv University**

Mathematics education researchers are always on the lookout for theories with predictive power – i.e., theories that enable the prediction of how students are likely to respond to given tasks. One such theory is the theory of the intuitive rules, which has been described in a series of articles. In this paper, we examined the predication that students will argue in line with the intuitive rule Same A–same B, that when increasing the size of two opposite sides of a square by a given factor and then reducing the size of the other two remaining sides by the same factor, the perimeter and area will remain the same. Our findings confirm that the reactions of students in various grades and with different levels of mathematics achievements to such tasks are indeed influenced by this intuitive rule.

In mathematics education, much like in any other domain, the contribution of a theory is measured by its explanatory and predictive powers. In the last decade the intuitive rules theory, which accounts for students' incorrect responses to many scientific and mathematical tasks has been developed and presented in several previous PME conferences (e.g., Stavy & Tirosh, 1994; Tsamir, Tirosh & Stavy, 1997; 1998; Tsamir, 1997). Currently, our efforts are devoted to examining its explanatory and predictive power. In this framework, the present study is aimed at examining the predictive power of the intuitive rule *Same A-same B.*

The intuitive rule *Same A-same B* (or *Equal A-equal B)* relates to comparison tasks. Typically, the students are asked to relate to two systems or two entities which are equal with respect to one quality or quantity ($A_1=A_2$), but may differ with regard to another quality or quantity, either $B_1=B_2$ or $B_1\neq B_2$. The students are asked to compare the two systems or entities with respect to quality or quantity B. It was found that students often claim that $B_1=B_2$ because $A_1=A_2$. Such deduction is frequently incorrect. One such example, presented in a previous PME paper is the claim that "quadrilaterals with equal sides (same A) have equal angles (same B)".

Mathematics and science education researches who have studied students' comprehension of specific notions have also found that often their responses are in line with this intuitive rule (e.g., Tsamir, Tirosh & Stavy 1997; 1998).

In the present article, we describe a study that predicted that students would argue, in line with the intuitive rule *Same A–same B,* that when increasing (adding / multiplying) two opposite sides of a quadrilateral by a given "amount" and reducing (subtracting / dividing) the same "amount" from the other two sides, [1] the **perimeter** of the original quadrilateral and the created one remain the same; and [2] the **area** of the original quadrilateral and the created one remain the same. We also wanted to investigate whether differences in students' grade levels or level of mathematics instruction affected the extent to which they applied the rule *Same A-same B.*

Method

Subjects

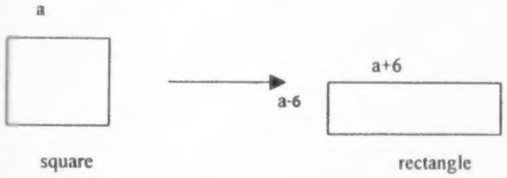
One hundred and seventy-one students in grades 9 to 11 from an urban school in Israel, having three levels of mathematics classes, participated in this study: 60 ninth graders, 57 tenth graders, and 54 eleventh graders. The participants belonged either to the lower level classes (LML) or to the classes of the mathematics majors (Advanced Mathematics Level – AML). About half of the participants from each grade level studied mathematics at the LML, while the other students were mathematics majors.

Materials and Procedure

A questionnaire consisting of 36 comparison tasks dealing with areas and perimeters of pairs of polygons, was distributed during a geometry lesson and the students were given about 90 minutes to answer it. Here we present four sample tasks. In each task, a rectangle is created when two opposite sides of a given square are lengthened by a given "amount" and the other sides of the square are shortened by the same "amount". The students were asked to compare the perimeters and the areas of the original square with those of the rectangle created. They were also asked to explain their answers in writing. Thirty students were afterwards individually interviewed orally, in order to better understand their lines of reasoning. The interviews were audio-taped, transcribed, and integrated into the 'Results' section.

Questionnaire

Consider a square, whose sides are "a" cm (a>6cm). A rectangle is created by lengthening two opposite sides of the square by 6 cm, and by shortening the other two sides by 6 cm.



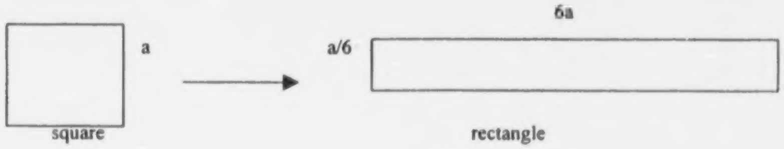
Problem 1

The Perimeter of the rectangle is -- larger than / equal to / smaller than / impossible to determine -- the perimeter of the square (circle your choice and explain your answer).

Problem 2

The Area of the rectangle is -- larger than / equal to / smaller than / impossible to determine -- the perimeter of the square (circle your choice, and explain your answer).

Consider the same square. A rectangle is created by multiplying the length of two opposite sides of the square by 6, and reducing the other two sides by the same factor, as described in the drawing.



Problem 3
The Perimeter of the rectangle is -- larger than / equal to / smaller than / impossible to determine -- the perimeter of the square (circle your choice, and explain your answer).

Problem 4
The Area of the rectangle is -- larger than / equal to / smaller than / impossible to determine -- the perimeter of the square (circle your choice, and explain your answer).

Basically, there were two kinds of tasks:

1. Tasks in which the responses are consistent with the intuitive rule *Same A-same B* (Problems 1 and 4): In Problem 1 the perimeter of the rectangle is equal to the perimeter of the square, and in Problem 4 the area of the rectangle is equal to the area of the original square. Thus, in these cases, the correct mathematical answer is in line with the intuitive rule *Same A-same B*. In problem 1: *Same A* (number of cm which was added to two sides and subtracted from the other two) - *same B* (perimeters). In problem 4: *Same A* (the factor by which two sides were multiplied and the other two were reduced) - *same B* (areas).

2. Tasks in which the responses run counter to the intuitive rule *Same A-same B* (Problems 2 and 3): In Problem 2 the area of the rectangle is smaller than the area of the original square, and in Problem 3 the perimeter of the rectangle is larger than the area of the original square. Thus, in these tasks the use of the rule *Same A-same B* leads to incorrect responses.

Results

The results are presented in two sections: the first deals with problems 1 and 4, where application of the intuitive rule *Same A-same B* leads to correct answers, and the second with problems 2 and 3, where it leads to incorrect responses.

I. The Intuitive Rule *Same A-same B* is Applicable

Most students from all grade levels (9, 10, and 11) and from both academic levels correctly claimed that **the perimeters** of the square and the rectangle were **equal** in the case of adding and subtracting six cm to opposite sides of the square (Table 1).

A somewhat lower percentage of students, but still a substantial number of participants from all grade levels and from both academic levels, argued correctly that **the area** of the square was **equal** to that of the rectangle when multiplying / dividing opposite sides of the square and decreasing the others by an equal factor

(Table 2). Three lines of reasoning accompanied the correct answers to problems 1 and 4, linking the equality of the "amount" added and reduced from the sides of the square with the equality of perimeters / areas. While the first expressed valid, formal arguments and the second was based on numeric, specific examples, the latter type of explanations, was in line with the intuitive rule *Same A–same B*.

Table 1: Students' responses, by grade and academic level to Problem 1 – addition and subtraction comparison-of-**perimeters** task (in %)

	Grade			Academic Level	
	9th	10th	11th	LML	AML
	(n=60)	(n=57)	(n=54)	(n=84)	(n=87)
The **Perimeter** of the Rectangle is:					
• Equal to the perimeter of the square	100	87	94	85	94
Larger than the perimeter of the square	--	3	6	7	3
Smaller than the perimeter of the square	--	10	--	8	3

Table 2: Students' responses, by grade and academic level to Problem 4 – multiplication and division comparison-of- **areas** task (in %)

	Grade			Academic Level	
	9th	10th	11th	LML	AML
	(n=60)	(n=57)	(n=54)	(n=84)	(n=87)
The **Area** of the Rectangle is:					
• Equal to the area of the square	82	79	91	71	90
Larger than the area of the square	18	21	5	21	10
Other answers	--	--	4	8	--

• correct answer
LML students in the lower mathematics level
AML mathematics majors

1. Formal arguments – In each of the cases (comparing the perimeters or the areas), about a third of the participants presented a formula for calculating the perimeter / area of each quadrilateral, showing that these expressions are equal. For instance, *the perimeters are equal because* $4a=4a+12-12$ (AML 9th grader); or *the areas are equal because* $a^2=6a\times\frac{a}{6}$ (AML 11th grader). These justifications, which drew upon geometrical theorems and relevant formulas, were mainly provided by 11th graders and AML students.

2. Numeric Example – In each of the two problems, a number of participants provided a specific, single, numeric example. They claimed, for instance, that, *if* $a=10$, *then* $a+6=16$ *and* $a-6=4$. *So, the perimeter of the square is* 4×10, *the perimeter of the rectangle is* $2\times(16+4)$, *and both equal 40. The perimeters are*

equal (AML 9th grader); or they claimed *For a=12, the area of the square is 12×12, the area of the rectangle is 72×2 both areas equal 144* (LML 11th grader). These justifications, which were based on familiarity with the relevant formulas, lacked any reference to a generalization that will go beyond the single, given example.

3. Arguments in line with the intuitive rule *Same A-same B* – About two thirds of the participants claimed that in the case of the perimeters, *we added 6 cm in one place and subtracted the same amount in another, this leaves us with the same perimeter* (AML 10th grader). In the case of the area, about a third of the students claimed that *we multiply and divide by the same number therefore the perimeter is the same* (AML 11th grader). Others incorrectly claimed that *the figures have equal perimeters therefore they also have equal areas* (LML 10th grader).

A number of students incorrectly claimed that the by changing the length of the sides we also **changed the perimeter / area** of the created rectangle. Most of these students claimed that the multiplication by 6 / addition of 6 cm was more significant than the corresponding division /subtraction. For example, *the longer side is more influential therefore the perimeter of the rectangle is larger than that of the square* (LML 9th grader).

Table 3: Students' responses, by grade and academic level to Problem 2 addition and subtraction comparison-of-**areas** task (in %)

	Grade			Academic Level	
	9th (n=60)	10th (n=57)	11th (n=54)	IML (n=84)	AML (n=87)
The **Area** of the Rectangle is:					
• Smaller than the area of the square	42	68	77	45	67
# Equal to the area of the square	46	12	16	49	14
Larger than the area of the square	12	15	3	--	16
Other answers	--	5	4	6	3

Table 4: Students' responses, by grade and academic level to Problem 3 multiplication and division comparison-of- **perimeters** task (in %)

	Grade			Academic Level	
	9th (n=60)	10th (n=57)	11th (n=54)	IML (n=84)	AML (n=87)
The **Perimeter** of the Rectangle is:					
• Larger than the perimeter of the square	56	74	84	60	80
# Equal to the perimeter of the square	37	14	12	28	16
Other answers	7	12	4	12	4

• correct answer # answer consistent with *Same A – same B*

LML students in the lower level AML mathematics majors

II. The Intuitive Rule *Same A-Same B* is Not Applicable

Table 3 shows that on the average, only about 60% of the students correctly claimed that when adding 6 cm to two opposite sides of a square and subtracting 6 cm from the other two sides **the area of the rectangle created is smaller than that of the original square**. Table 4 shows that on the average about 70% of the students correctly claimed that when multiplying two opposite sides of a square by six and the other two sides by a one-sixth of their lengths **the perimeter of the rectangle created is larger than that of the original square**.

Three types of justifications were given to the correct responses:

1. Formal arguments – In each of the cases (comparing the perimeters or the areas), about a third of the participants provided formula for calculating the perimeter / area of each quadrilateral, showing that these expressions are not equal. For instance, *the area of the square is larger because* $a^2>a^2-36$ (AML 11th grader); or *the perimeter of the square is smaller:* $4a<2\times 6a+2\times \frac{a}{6}$ (LML 10th grader). These justifications, which drew upon the relevant formulas, were mainly provided by 11th graders and AML students.

2. Numeric Example – In each of the cases (comparing the perimeters or the areas), a number of 10th and 11th grade LML students substituted a single number to obtain a solution. They claimed, for instance, that, $a=10$, *so the area of the square is* 100, $100>100-36=64$, *therefore the area of the square is larger* (LML 10th grader). These justifications were based on the relevant formulas with no generalization.

On the average, about a third of the students wrongly responded that when two opposite sides of a square are prolonged by 6 cm and the other two sides are shortened by an equal number of cm, than the **area of the given square is equal to that of the resulting rectangle.** In the same spirit, about a quarter of the participants wrongly argued that when two opposite sides of a square are multiplied by 6 and the other two sides are divided by the same factor, then the given square and the resulting rectangle have **the same perimeters**. Justifications to these assertions were usually in line with the intuitive rule *Same A - same B,* and included the following variations: [1] The most prevalent were arguments, such as, *the same number was transferred from one place to another, therefore there is no change in the perimeter and the areas, it remains the same* (9th grade, LML); [2] A number of AML 11th graders strangely argued that *squares and rectangles have equal angles, therefore they have equal areas and equal perimeters.* One of them elaborated on his explanation saying that *both are parallelograms, meaning that the area is* $a\times b\times \sin\alpha$. *The sum of a and b is the same,* α *is the same, so the area is the same;* [3] Several LML 9th graders claimed that since one (either perimeter or area) is equal the other must also be equal.

It is notable that while in all cases the correspondence between grade level and correctness of the responses was found to be non-significant, the correspondence between academic level and correctness of responses was significant (problem 2, area: $x^2=19.16$, $DF=4$, $p<.01$; problem 3, perimeter: $x^2=12.25$, $DF=4$, $p<.05$; problem 4, area $x^2=12.16$, $DF=4$, $p<.05$). The tendency to provide correct answers to each of the three tasks was significantly higher among the AML students than among the LML ones.

Discussion and Educational Implications

The findings of this study support our prediction that students at various grade levels would link the area or perimeter of a square, wherein two sides had been elongated and two sides reduced by an equal "amount", with the area or perimeter of the resulting rectangle, and declare them to be equal. While this type of reasoning may be explained by Piaget's theory about the use of compensation to attain conservation (e.g., Piaget & Inhelder, 1963; 1974), Tirosh and Stavy viewed such responses as an instance of the intuitive rule *Same A-same B* (Stavy & Tirosh, in press).

The application of this rule in the cases of problem 2 (areas, when adding/subtracting 6 cm) and problem 3 (perimeters, when multiplying/dividing by 6) led a substantial number of students to erroneous conclusions – (a) the areas of the original square and the created rectangle were incorrectly declared to be *equal*; and (b) the perimeters of the original square and the created rectangle were incorrectly regarded as being *equal* as well. This phenomenon was quite prevalent among the younger students and LML students, but found also among about 15% of the mathematics majors, most of whom explicitly argued *the same-the same*. The older, mathematics majors supported their claims that *the same* (added and taken away)-*the same* (area) with formal arguments, also overgeneralizing irrelevant mathematical theorems (such as the area of a parallelogram). Other students presented their claims in a rather general manner. The result was the use of invalid, formal-looking justifications for the answers which were probably determined by the intuitive rule *Same A - same B* (see also Tsamir, Tirosh and Stavy, 1998).

The intuitive rule *"Same A-same B"* was found to direct students' responses. On the average at least 25% of the students erred when applying the intuitive rule to obtain solutions. It should be noted that the problems chosen for discussion here dealt with squares and rectangles – figures children become familiar with in the lower grades of elementary school and which are subsequently used and re-used at all levels. The fact that even with such problems, secondary school (even AML) students' responses were influenced by the intuitive rule, shows how pervasive this influence is. (In the oral presentation we will provide additional examples from our study to support this assertion).

Correct answers to problems 1 and 4 could be based upon either the relevant mathematical theorems, or the application of the intuitive rule *Same A–same B*. Hence, even when students did get the correct answers to the problems and validated them with claims, such as, *Equal sizes added and taken away correspond to equal perimeters* (LML 9th grader), we cannot be certain that their responses were attained entirely via formal knowledge. These students may have reached the correct conclusion by applying the intuitive idea of *same* (added and taken away)- *same* (area), or *same* (perimeter)- *same* (area). We must conclude that correct answers which are consistent with an intuitive rule do not necessarily reflect students' correct understanding.

In light of our findings, we recommend that teachers be aware of the role which this intuitive rule plays in students' analysis of problems and their solutions. Furthermore, when presenting problems, teachers should consider whether their solutions may be in line with an intuitive rule or counter to it. When presenting problems which lend themselves to an intuitive solution, teachers should not be satisfied with the correct answers alone, but probe further to be certain that the students are not just applying the intuitive rule. To sum up, based upon our experience regarding students' ways of applying this rule, we can predict their problem-dependent, correct as well as erroneous, responses. Such an ability to foresee possible intuitive triggers and obstacles, should serve as a tool for meaningful instruction. The oral presentation will refer to suggestions and implications for the design and construction of such instructional tools.

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INTUITIVE BELIEFS AND UNDEFINED OPERATIONS: THE CASES OF DIVISION BY ZERO

Pessia Tsamir Dina Tirosh
Tel-Aviv University

This paper describes a study that explores secondary school students' conceptions of division by zero. Our aims were: (1) to examine whether secondary school students identify expressions involving division by zero as undefined, or tend to assign numerical values to them, (2) to study their justifications, and (3) to analyze the effects of age (grade) and level of achievement in mathematics on responses. Our findings indicate that a substantial number of the participating students argued, in line with the numeric-answer belief, that division by zero results in a number. Moreover, performance on division-by-zero tasks did not improve with age. Level of achievement in mathematics, however, was highly related to performance on tasks. Possible causes and the educational implications of these findings are discussed.

Intuitive beliefs have the characteristics of intuitive thinking: Self evidence, intrinsic certainty, perseverance, globality and coerciveness (Fischbein, 1987). In this paper we present our initial attempts to study one intuitive belief about mathematical operations: the numeric-answer belief (i.e., an intuitive belief that every mathematical operation must result in a numeric answer).

The mathematical experiences of many children during their first years of schooling is primarily comprised of performing manipulations and arriving at numerical solutions. Such extensive experience with mathematical operations inevitably leads to a development of a numeric-answer belief. Division by zero is usually the first undefined mathematical operation that students encounter during their school studies. Clearly, the mere existence of an undefined mathematical term violates the intuitive, numeric-answer belief. Adherence to this belief might result in assigning numerical values to expressions involving division by zero.

A tendency to claim that division by zero results in a number was indeed found among elementary and middle school students, prospective teachers and teachers (Ball, 1990; Blake & Verhille, 1985; Grouws & Reys 1975; Reys, 1974; Tsamir, 1996a; 1996b; Wheeler & Feghali, 1983). Surprisingly, we found no study that examined secondary school students' responses to tasks involving division by zero. Such inquiry could extend our understanding of the relationship between intuitive beliefs about mathematical operations and students' actual, incompatible practices, as students in secondary schools are often expected to apply their knowledge about division by zero in various situations.

This paper describes a study that explores secondary school students' conceptions of division by zero. Our aims were: (1) to examine whether students in secondary schools identify expressions involving division by zero as undefined or tend to assign to them numerical values, (2) to study their justifications, and (3) to analyze the effects of age (grade) and level of achievement in mathematics on students' responses to such expressions.

Method

One hundred and fifty-three students from Grades 9, 10, and 11 in a secondary urban school in Israel participated in this study. About half of the students in each grade level studied Advanced Mathematics Level (AML) while the other half studied Low Mathematics Level (LML). A written questionnaire including 20 multiplication and division expressions was administered to the participants during a mathematics class session of about 60 minutes. Subjects were asked to read each expression, to give a numeric solution, if possible, or to explain why it is impossible to provide a numeric solution. The questionnaire included 11 expressions involving division and zero, seven of which were division by zero expressions. The following five categories of expressions were included in the questionnaire:

1. Four division expressions of the type "$a \div 0$ ($a \neq 0$)" (i.e., $12 \div 0$, $\frac{14}{0}$, $\frac{6}{3-3}$, $\frac{4}{0}$).
2. Three division expressions of the type "$0 \div 0$" (i.e., $0 \div 0$, $\frac{4 \cdot 0 + 3 \cdot 0}{0}$, $\frac{0 \cdot (2+3)}{0}$).
3. Seven division expressions of the type "$a \div b$ ($a \neq 0$, $b \neq 0$)" (i.e., $\frac{4-0}{0-4}$, $\frac{16-4}{4}$, $9 \div 3$, $\frac{15}{3}$, $\frac{1+0}{0-1}$, $\frac{18}{6}$, $\frac{7-1}{6:2}$).
4. Four division expressions of the type "$0 \div a$ ($a \neq 0$)" (i.e., $0 \div 6$, $0 \div 16$, $\frac{0 \cdot 4}{3}$, $\frac{0}{10}$).
5. Two multiplication expressions of the type "$a \cdot 0$" (i.e., $0 \cdot 0$, $4 \cdot 0$).

We were mainly interested in students' responses to the seven expressions involving division by zero (Categories 1 and 2). The other 13 expressions were designed to serve several purposes: (a) to mix the undefined division expressions with defined division expressions, thereby reducing the chance of receiving automatic, "undefined" responses, (b) to ensure that poor performance on division by zero expressions is not a result of a lack of competence in computing multiplication and division expressions, and (c) to record overgeneralizations of "undefined" responses to multiplication and division expressions involving zero. Past studies indicated that a substantial number of elementary school students overgeneralized their undefined responses to all division expressions involving zero (e.g., Grouws & Reys, 1975). The expressions included in Categories 4 and 5 could trace similar overgeneralized responses among secondary school students.

The division expressions in the questionnaire were written in two standard division notations: a ÷ b and $\frac{a}{b}$. Past studies indicated that elementary school students performed better with the a÷b notation (Grouws & Reys, 1975). We were interested in examining whether differences in notation have similar impact on secondary school student performance.

Typically, subjects elaborated on their responses to the questionnaires and thus provided substantial information about their reasoning. Still, in some cases, follow-up interviews in which participants were encouraged to further explain their responses were needed. In these cases, the interviewee's responses were added to the original questionnaire, providing a fuller picture of his or her related reasoning.

Results

In this section, we present the findings regarding each of the two different cases of division by zero. Before reporting these results, the following comments seem in order:

1. No substantial differences were found between participants' performance on the two different notations of the division expressions ($\frac{a}{b}$ and a ÷ b). Hence, these two types of expressions will be treated together.
2. Almost all participants responded correctly to division expressions that do not involve zero (Category 3- 99% correct on average). Hence, inadequate performance on division involving zero expressions could not be attributed to lack of competence in solving division expressions.
3. All students correctly solved the multiplication expressions involving zero (Category 5).
4. Most students knew that $0 \div a = 0$ for $a \neq 0$. We can conclude that overgeneralization of undefined responses was infrequent among secondary school students (though the typical incorrect response was "undefined").

It should also be noted that no substantial differences were found between the responses and the justifications of students in different grade levels. However, the nature of justifications given by AML and LML students were radically different.

Division of a non-zero number by zero

The majority of the participants (74%) responded that division of a non-zero number by zero is undefined (see Table 1). The most common incorrect responses were either zero or the dividend. Another incorrect response, "∞", was given by 12% of the AML 11th graders. The way they wrote their solutions (e.g., $12 \div 0 = \infty$), and the fact that they provided no justifications to their responses suggested that they regarded ∞ as a specific, numeric answer.

AML students used the following, three justifications to explain their "undefined" responses:

1. Relying on the definition of division as the inverse of multiplication. Most AML participants explained that any definition of expressions of the type a ÷ 0 for a 0 would violate the definition of division as the inverse of multiplication. Some noted that such a definition would violate either the definition of division as the inverse of multiplication or the theorem $c \cdot 0 = 0$ for every c. This mathematically-based justification is often used by high-school teachers to explain why division of a non-zero number by zero is undefined.

Table 1 Correct responses to a÷0 (a≠0) expressions by grade and level of mathematics achievement (in %)

	Grades			**Level of Achievement**		
Expression	9 n=39	10 n=58	11 n=56	3 n=73	5 n=80	**Total** **n=153**
12÷0	77	72	71	66	80	**74**
$\frac{14}{0}$	87	72	70	67	83	**76**
$\frac{6}{3-3}$	77	67	69	57	81	**70**
4÷0	82	76	76	69	85	**77**
Average	**81**	**72**	**72**	**65**	**82**	**74**

2. Applying a notion of limit. The second, mathematically-based justification of the undefined responses consisted of applying a notion of the limit of a ÷ x as x tends to zero through positive and through negative numbers. A typical reaction of this type was $12 \div 6 = 2$, $12 \div 3 = 4$, $12 \div 1 = 12$, $12 \div \frac{1}{2} = 24$, $12 \div \frac{1}{4} = 48$, $12 \div \frac{1}{16} = 192$… as I get closer and closer to zero, the numbers increase. Now, I'll do the same, but this time I'll approach zero from the left side of the number line. I have: $12 \div (-12) = (-1)$, $12 \div (-6) = (-2)$, $12 \div (-3) = (-4)$, $12 \div (-1) = (-12)$,

$12 \div (-\frac{1}{2}) = (-24)$, $12 \div (-\frac{1}{4}) = (-48)$, $12 \div (-\frac{1}{16}) = (-192)$. The numbers decrease. So, there is a jump at the point zero and it is impossible to find a number for $12 \div 0$". This justification was provided by 16% of the AML students in Grade 11.

3. Using the compromised, "∞ - undefined" notion. Another explanation, suggested by several AML students (13%, 10% and 16% in grades 9, 10 and 11 respectively) was that "division by zero is undefined because it is infinity, and infinity is undefined". Interviews with students who used this justification revealed that for them division by zero results in the number ∞. This number (∞) was undefined either because its exact location on the number line was unknown, or because its value was not fixed. It should be noted that this "∞ - undefined" response does not contradict the numeric-answer belief.

Students doing LML used one, common justification to explain their "undefined" responses:

1. Illustrating that division of a non-zero number by zero is, in practice, impossible. Participants who used this justification related to division as "sharing" and to zero as "nothing". A typical response was "$12 \div 0$ means sharing 12 cookies in equal parts among no kids. It is impossible to share 12 cookies among no kids, therefore $12 \div 0$ is meaningless, undefined".

Division of zero by zero

About 60% of all the participants correctly argued that $0 \div 0$ is undefined (see Table 2). The only incorrect response to these tasks, zero, was given by all other participants.

Table 2 Correct responses to 0÷0 expressions by age and level of mathematics achievement (in %)

	Grades			**Level of Achievement**		
Expression	9 n=39	10 n=58	11 n=56	3 n=73	5 n=80	**Total n=153**
$0 \div 0$	62	53	61	46	70	**56**
$\frac{4\cdot 0+3\cdot 0}{0}$	64	60	66	52	73	**60**
$\frac{0(2+3)}{0}$	64	63	69	55	75	**63**
Average	**63**	**58**	**65**	**51**	**73**	**60**

Two justifications were given by AML students to explain their "undefined" responses:

1. Viewing 0 ÷ 0 as a specific instance of division by zero. Most AML students explained that division by zero is undefined for any number, including zero. In an interview with one of the participants who provided such a justification, we asked if $0 \div 0$ could also be regarded as a specific instance of $a \div 0$ for $a \neq 0$. This participant replied that "in principle, it is possible, but I know that $0 \div 0$ is a specific instance of $a \div 0$. I do not know why".

2. Relying on the single-value requirement. Few AML students specified the that an operation should fulfill the single-value requirement (10%, 9%, and 12% in grades 9, 10 and 11 respectively). A typical explanation of this kind was" "$0 \div 0$ is undefined because if $0 \div 0 = x$ then $x \cdot 0=0$. But this is true for every number".

The LML students used one, common justification to explain their "undefined" responses:

Illustrating that division of zero by zero is, in practice, impossible. Most LML students argued that practically it is impossible (meaningless) to divide zero by zero. A typical response was provided by Gil (10th Grade):

Gil: 0 ÷ 0 means sharing 0 candies in equal parts among no kids. It is impossible to share 0 cookies among no kids, and therefore 0 ÷ 0 is meaningless.

Conclusions and Educational Implications

Our study explored how secondary school students, who were expected to apply their knowledge about division by zero in various situations, solved the tension between the intuitive, numeric-answer belief and the formal, mathematical decision not to define these expressions. The findings indicate that about a third of the participating secondary school students argued, in line with the numeric-answer belief, that division of a non-zero number by zero results in a number (the dividend, zero, or the number infinity). The claim that $0 \div 0 = 0$ was even more frequent (40% of the students came up with this response). The justifications of students who correctly judged division by zero to be undefined revealed that the apparent contradiction between the intuitive, numeric answer belief (i.e., every drill results in a numeric answer) and the adequate answer (i.e., division by zero is undefined and thus it is not a number) was not obvious to all of them. Some "reconciled" these standpoints by arguing that division by zero is undefined because the result is "∞ - undefined". A different, compromise solution was to use the notions "practically meaningless, impossible to perform" as synonyms to "undefined". Such responses ("an undefined number", "impossible to perform, meaningless") allow their users to provide seemingly correct responses while avoiding crucial questions related to division by zero, e.g., "Why division-by-zero expressions do not result in a number?" or "Why division-by-zero expressions are doomed to be undefined?".

Our data also show that secondary school students' performance on division-by-zero tasks did not improve with age (grade level). Level of mathematics achievement, however, was strongly related to task performance. Striking differences were observed between high and low achieving students' performance on, and justifications for expressions involving division by zero. Most high-achieving students used mathematically-based arguments to justify their responses (e.g., related to division as the inverse of multiplication, or applied an intuitive notion of limit). The low achievers, however, relied on practical, concrete justifications, interpreting division as sharing and zero as nothing. The sharp split in the nature of justifications provided by AML and LML students deserves consideration. It seems that these two groups developed different sociomathematical norms related to what counts as an acceptable mathematical justification (Yackel & Cobb, 1996).

The reliance of LML students on the sharing model of division could be used as an argument against applying practical models to show the impossibility of division by zero. Clearly, it is possible to reason for (and against) the use of practical models in elementary schools with students whose acquaintance with operations is limited to non-negative, whole numbers. However, it seems that no one would argue against the need to re-evaluate the applicability of such models, if they are indeed to be used, when the operation of division is extended to fractions and to negative numbers. It is essential that teachers encourage their students to reflect on the types of explanations they use and be aware of their realms of application. Such examination could then lead to an exploration of some major issues related to intuitive beliefs, formal definitions, mathematically-based and intuitively-based justifications and mathematical operations (i.e., How do mathematicians make decisions about definitions of operations? What are the main properties of mathematical definitions? What are the main properties of mathematical operations? What are the reasons behind choosing a certain definition? Is there a general policy according to which mathematical operations are doomed to be either defined or undefined?). Such discussions could assist teachers in their attempts to lessen the undesirable myth that mathematics is about memorizing unreasonable rules and to promote a view of mathematics as a human-made, reasonable discipline.

Coming back to a main issue discussed in this paper, that of the relationship between intuitive beliefs and undefined operations, we would argue that the intuitive, numeric-answer belief could affect students' responses not only to tasks involving division by zero (e.g., finding excluded values of rational equations) and to tasks involving other, undefined mathematical terms but also to other, seemingly different situations (e.g., simplifying algebraic expressions). Given this state of affairs, a teacher could naturally ask: "How can I help my students overcome the coercive effect of the numeric-answer belief?" The teacher's task is indeed complicated. Fischbein (1987) suggested that a major aim of mathematics education is to raise students' awareness of the role of intuitions in their thinking processes and to develop their ability to analyze and control them. In the case of division by zero, it is important to explicitly relate, in class, to the intuitive belief that every mathematical operation must result in a numerical answer, to discuss its possible sources and to demonstrate its impacts on our reasoning processes. The teacher could refer to the observed differences in students' performances in the two cases involving division by zero ($a \div 0$ for $a \neq 0$, and $0 \div 0$), drawing on the profound mathematical and psychological differences between these cases. Other common intuitive beliefs about mathematical operations could be addressed as well (e.g., addition and multiplication makes bigger, division makes smaller), leading to a more comprehensive discussion on the differences between intuitive beliefs about and formal definitions of mathematical operations.

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FACTORS CONTRIBUTING TO LEARNING OF CALCULUS

Behiye Ubuz and Burcu Kırkpınar

Middle East Technical University, Ankara, TR

This paper reports on a study of students learning the concepts of calculus, particularly derivative, and factors affecting learning in undergraduate calculus course in a computer-based learning environment. Interactive Set Language (ISETL) was used to help students to construct mathematical concepts on a computer, followed by the discussion held in the classroom. DERIVE was also used to do manipulations and draw graphs. The results showed that there was a significant improvement in learning derivative concepts in general and in two dimensions (graphical interpretation and the use of the definition of the derivative). In addition, pre-test score was the best predictor of success on the post-test score and on the post-test score calculated on the problems related to the graphical interpretation. Department, study major of the students, was the best predictor on the post-test score calculated on the problems related to the use of the definition of derivative.

Introduction

The study reported in this paper is part of a comprehensive research concerning students' learning of calculus concepts. In this paper, the results of a quantitative study concerning the effect of an instructional treatment, based on having students make various constructions on the computer using ISETL (Dautermann, 1992) and develop manipulative skills and visualization using DERIVE (1989), followed by classroom discussion of mathematics concepts corresponding to these computer tasks, on the learning of derivative and the factors effecting learning are reported. There was also a certain amount of paper - and – pencil work for the students to do, both in and out of class. The purpose of this study was to answer the following research questions.

1. Is there a significant improvement (at the 0.05 level) in learning derivative concepts through the computer-based learning environment?
2. Which of the internal factors of the students, students' personal attributes (gender, age, department, university entrance examination scores, and pre-test scores), and external factor (teacher) as independent variables has a significant effect on the success in the post-test? Is this influence in a positive or negative direction?
3. To what extent do the variables found significant (at the 0.05 level) in question 2 predict a student's score on the post-test?

A theoretical framework

There have been studies (e.g Breindenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky, 1997) concerning teaching and learning of mathematical concepts using ISETL since the development of the programming language SETL (Schwartz, Dewar, Dubinsky, &Schonberg, 1986). The researchers have mainly focused on the development and understanding of certain concepts.

Many educators believe that the relationship between the internal and the external factors related to the students and their effect on learning is critical. For example, it was noted that students' pre-instructional knowledge (Carpenter & Fennema, 1991; Ferrini-Mundy & Lauten, 1994; Skemp, 1971) and gender (George, 1999; Fennema & Hart, 1994) might influence learning.

In this study, we are particularly interested in the effect of the computer-based learning environment on learning derivative concepts and what other factors are effecting learning besides the treatment.

Method

Subjects

The sample consists of 59 first year undergraduate students in four sections of Math 153 Calculus I course offered at Middle East Technical University during fall 1996– 1997. Students were pursuing a major either in mathematics or mathematics education. The sections were formed randomly and different teachers taught each section. Two of those teachers who taught section 1 and 2 were male and the rest were female.

Table 1 shows the numbers of students, who took the pre-test and the post-test on derivative. The students who took both the pre-test and the post-test were taken as the sample of the study.

Table 1: The sample of the study

Section	Pre-test	Post-test	Pre-test ∩ Post-test
1	21	25	17
2	15	17	13
3	26	18	15
4	26	15	14
Total	88	75	59

33 (%56) students of those 59 were majoring in mathematics and the rest 26 (%44) students in mathematics education. 57 (%97) of those students have not taken this Math 153 course before and 53 (%90) students have also not taken Math 100 course given prior to Math 153 course. Math 100 course is given to the students who are not able to do 35 mathematics questions out of 52 in the university entrance examination. In the sample, 20 (%34) students were female and 39 (%66) students were male.

Instrument

The test used for assessing students learning of derivative consisted of 6 problems, some of which having different tasks (altogether 32 tasks), on which students were to work individually to provide written responses (see Appendix). Demographic survey questions to gather personal information about each student were included at the beginning of the test. The test was given as a pre-test and post-test without prior warning. The pre-test was administered at the beginning of the semester and the post-test at the end of the semester. Each semester lasts 14 weeks. Each task in the problems were graded by one of the four categories: correct (3), partially correct (2), incorrect(1), and missing (0). The scoring criteria for each task together with the examples of responses and the distribution of the number of students are not given in here due to the space problem.

The factor analysis carried out for the problems in the pre-test revealed that the test was two dimensional. The first factor was related to the graphical interpretation (GI) (problems 1, 2, 4, and 6) and the other was related to the use of the definition of derivative (DfD) (problems 3, and 5).

Treatment

The study was conducted in a course (Math 153) designed to teach functions, limit, derivative of a function, graph sketching, problems of extrema, and basic theorems of

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differential calculus: intermediate, extreme, and mean value theorems. The instructional treatment consisted of mainly having students make various constructions on the computer using the programming language ISETL, followed by class discussion of concepts corresponding to these computer tasks. DERIVE was also used by the students for doing activities which are difficult to do by hand. For example, drawing the graph of ($\mathrm{Sin}\frac{1}{x}$). There were also exercises to be done with pencil and paper after the class. Handouts were given on how to use DERIVE and ISETL at the beginning of the course. The textbook used in the course was *Calculus, Concepts, and Computers* (Dubinsky, Schwingendorf, & Mathews, 1995).

Classes met 6 class hours of a week for 50 minutes each. Two of these hours were at the computer laboratory. There were two 2-class hour sessions during the week and students had to attend only one of these sessions Some weeks, classes met in the class instead of computer laboratory, and quiz was given each such week. In the lab, students worked individually, each with her or his own terminal. Assistants were available to answer questions, give help with syntax, and etc. There were three computer rooms available, each equipped with 20 computers.

The first week of the semester was used to form the groups of 4 students and to make the introduction for the course. Students who knew and agreed with each other, and had common free time included in the same group. Each week groups were required to complete one activity on the computer by submitting it on the disk, and to complete exercises done with pencil and paper. The group members sat together in the class, because often they had to answer the questions collectively. Every member of each group must be involved in these works as they were going to take their exams individually. Late submissions were not accepted since solutions to the assignments were discussed in class.

The main purpose of the lab sessions was to make sure that every student had at least attempted to perform certain computer tasks before coming to class. The idea was to present the students with the problems so that they could make useful mental constructions. Brief explanations of the activities together with their examples are given below:

I. Functions

1. Writing computer programs of the given different situations where the functions are given in the form of: piecewise , graph, (in)finite SMAP , table, tuple, and string. For example, see question 1 in the book called *Calculus, Concepts and Computers* (CCC) (Dubinsky et al., 1995, p.69). This question is an example of the type piecewisely defined function.
2. Interorizing the action by taking different values from the domain and evaluating them. This makes the students to think about what computer is doing when it makes those evaluations. For example, see the question 1 in the CCC.
3. Drawing the graph of given expressions to understand the function concept and to learn the graph reading.
4. Encapsulating the composition of functions by giving an ISETL code directly and then make students to give meaning to the code. For example, see question 3 in the CCC (p.80).

II. Limit

1. Understanding that the limit value exists regardless of the existence of the function value at that point. For example, question 2 in the CCC (p.132).
2. Interorizing the behaviour of a function near a specified point or at large values i.e. variable tends to infinity. For example, question 3 in the CCC (p.132).

3. Making the idea of the formal definition of the limit more concrete by writing a computer function for taking limit , right limit , left limit , limit at infinity and limit at minus infinity. For example, question 1 in the CCC (p.142).

III. Derivative

1. Encapsulating the concept of derivative by the help of writing a computer program using the concepts difference quotient and the limit. For example, question 1 in the CCC (p.191).
2. Determining the extreme values of a function by graph reading. For example, question 7 in the CCC (p.219).

In the course there were 2 midterms and one final exam. These were in the form of solving problems or proving with paper and pencil without calculator or computer. Exams also contained short questions to be solved using the computer language ISETL. Grading was as listed: Assignments (activities and exercises) 10 %, Class work (participation in class, quizzes, and attendance) 20 %, 2 midterm exams 50 %, Final exam 40 %.

Results

An initial question involved investigating whether there is a significant mean difference between the pre-test and the post-test scores of the students, and between the pre-test and the post-test scores on the problems related to GI and DfD. T-tests for paired data were performed to answer these questions. The comparison of the means (see Table 2) in the pre-test and the post-test indicated that there is a statistically significant mean difference (t=5.62, df=58, p=0.000). The comparison of the means in the pre-test and the post-test on the problems related to GI and DfD showed that there is a statistically significant mean difference on the GI (t=5.54, df=58, p=0.000) and DfD problems (t=2.40, df=58, p=0.02). Although, there is a statistiscally significant mean difference on both type of problems, the mean difference on DfD problems is not practically significant (effect size is equal to 0.36).

The relationship between success on the post-test and other variables was investigated by using a stepwise multiple linear regression analysis. A stepwise multiple linear regression analysis was performed with the post-test score and the post-test score calculated on the problems related to GI and DfD as the dependent variables. Independent variables were gender, age, department, university entrance exam score and pre-test score of the students, and teacher difference. Gender was coded 1 for female and 2 for male. Department was coded 1 for mathematics and 2 for mathematics education. Teacher difference was coded as 1, 2, 3, and 4 according to the sequence given in Table 1. While university examination score ranged from 530 to 604, age ranged from 17 to 22.

Results of the multiple regression analysis to determine variables that relate to success on the post-test are given in Table 3. The subjects' scores on the pre-test which accounted for 10% of the variation in the post-test, entered the equation first. As seen in the correlation matrix in Table 2, pre-test score was positively correlated to post-test score (.32). Teacher entered the equation next, and together with the pre-test scores they accounted for 19% of the variation in the post-test scores. The last variable, department, increased the R^2 so that together with the first two variables they accounted for 29% of the variation in the post-test scores. As seen in Table 2, mathematics students were more successful on the post-test than the mathematics education students (-.24).

Results of the multiple regression analysis to determine variables that relate to success on GI problems in post-test are given in Table 4. The subjects' scores on the pre-test which accounted for 10% of the variation in the GI problem scores in the post-test, entered the

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equation first. Teacher entered the equation next together with pre-test scores and they accounted for 18% of the variation in the post-test scores on GI problems.

Table 2: Means, standard deviations, and correlations of all variables

Variable	1	2	3	4	5	6	7	8	9	10	M	SD
1												
2	-.09											
3	-.06	.21										
4	.11	.22	-.01								18.6	.94
5	-.12	.21	.33*	-.09							570.	12.9
6	.09	.01	.00	-.08	.25						69.9	13.4
7	-.10	-.24	.22	-.08	.13	.32*					80.1	9.9
8	-.10	.00	.04	-.01	.12	.49**	.10				12	2.6
9	.12	.01	-.01	-.09	.24	.98**	.32*	.33*			57.9	12.4
10	-.06	-.18	.21	-.09	.09	.32*	.98*	.07	.33*		67.1	9.3
11	-.18	-.32*	.13	.05	.21	.07	.37**	.14	.05	.16	12.9	2.2

* $p<0.05$, ** $p<0.01$

1=Sex, 2=Department, 3=Teacher, 4=Age

5=University Exam Score, 6=Pre-test Score, 7=Post-test score

8=Pre-test score on the problems related to the use of definition of the derivative

9= Pre-test score on the problems related to the grapical interpretation

10= Post-test score on the problems related to the graphical interpretation

11= Post-test score on the problems related to the use of definition of the derivative

Table 3: Stepwise multiple regression analysis to predict post-test scores

Prediction Variable		R	R^2	R^2 Change	F Change
1	Pre-test Score	.32	.10	.10	6.10
2	Teacher	.44	.19	.09	6.01
3	Department	.54	.29	.10	7.31
Variables not entered (F to enter less than 3.84)					
4	Age				
5	University Exam Score				
6	Sex				

Table 4: Stepwise multiple regression analysis to predict post-test scores on the GI problems

Prediction	Variable	R	R^2	R^2 Change	F Change
1	Pre-test Score	.32	.10	.10	5.99
2	Teacher	.42	.18	.07	4.68
Variables not entered (F to enter less than 3.84)					
3	Department				
4	Age				
5	University Exam				
6	Sex				

Results of the multiple regression analysis to determine variables that relate to success on DfD problems in the post-test are given in Table 5. Department which accounted for 11% of the variation in the post-test scores related to the DfD problems. University entrance examination scores entered the equation next, and together with department they accounted for 19% of the variation in the post-test scores related to the DfD problems. The last variable, teacher, increased the R^2 so that together with the first two variables they accounted for 23% of the variation in the post-test scores on DfD.

Table 5: Stepwise multiple regression analysis to predict post-test score on the problems related to the DfD

Prediction Variable	R	R^2	R^2 Change	F Change
1 Department	.33	.11	.11	6.67
2 University Exam	.44	.19	.08	5.36
3 Teacher	.48	.23	.04	2.75
Variables not entered (F to enter less than 3.84)				
4 Sex				
5 Pre-Test				

Conclusion

It seems reasonable to conclude from these results that when the computer-based approach described here was used, a significant improvement in learning the derivative concepts can occur. This result was consistent with the previous research in which the similar approach used (Breindenbach et al.,1992; Dubinsky, 1997). The improvement of the students on the GI problems was more than on the DfD. Although the overall improvement of the students on the problems related to the DfD was weaker, some learning seems to have taken place. As seen in Table 2, the mean on the pre-test related to the problems on DfD was 12 and the maximum score on this part was 15. It does appear that students had already learned to use the definition of the derivative on this type of problems. These problems requires operational procedures by using the quotient formula and these procedures take important part in the school curriculum and the university entrance examination.

To understand the effect of the treatment, it is also important to understand what other factors influence the learning. A variable highly related to the success on the post-test was the pre-test. This was consistent with the previous studies (Carpenter & Fennema, 1991; Ferrini-Mundy & Lauten, 1994; Skemp, 1971). This variable was positively correlated to the post-test score, in general, and the post-test score on the problems related to the GI. However, the pre-test was not the predictor of the success in the problems related to DfD. Besides the pre-test, the other variable that increased the predictability of success on the post-test in general and on the post-test in two dimension was teacher. Even teacher as a variable seems to be a factor in the prediction of success on the post-test, but teacher effect may be subject to many influences that were not examined such as the motivation of the students or attitudes towards computers and the treatment. Other variable related to success in the post-test, in general, and on the problems related DfD was the department. This variable was negatively correlated with the post-test scores. This means that mathematics students were more succesful than the mathematics education students. It may be that mathematics students were more motivated to learn mathematics in the computer-based learning environment. The department was not the variable to predict the success on the problems related to the GI. As the treatment involved activites related to the GI and occurred many times in different topics, both mathematics and mathematics education students showed the same success on the post-test. The university examination score was the predictor only for the success on the problems related to the DfD. The reason might be that students who had high university examination scores had already solved this type of problems in the university examination.

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The variables found here as factors predicting success can be used to determine which students may benefit from treatments designed to improve students' learning and to modify existing classroom instruction to improve all students' learning.

The comments taken from the students after the treatment indicated what changes can be made to improve the instructional treatment. It seems that students did not realise how they could learn in such an envoriment. The lack of knowledge on how to use the computer and the software, and not being used to this kind of approach all through their education till the university, prevented them to get more benefit from this treatment. Besides, while discussing the questions given in the assignments in the class, the class teacher posed each question to the class to discuss on it and then get the answer from a person in one group. When the question answered correctly, the answer assumed to be understood by everyone in the class. Instead of doing this, it seems better to expect an answer from each group and attention should be given to get the answer each time from a different person. However, the difficulty in using the cooperative learning properly is understandable as there were a lot of important topics to be covered in a certain period of time.

This research rises two questions for further research. Firstly, are the factors affecting learning of calculus through computers also the factors affecting the learning in the traditional treatment? Which one of the teacher related factors such as attitudes toward computers, personal beliefs about the ways students should be taught and students related factors such as motivation and attitudes toward the computers are the ones affecting students'learning?

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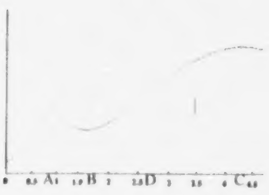
Appendix (The test on Derivative)

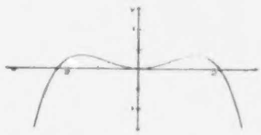
1



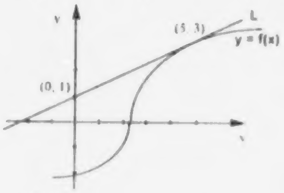
Above is the graph of the function *V*, whose domain is the interval [0, 5]

(a) For which value(s) of x is the function increasing?
(b) For which value(s) of x does *V* has a local maximum?
(c) For which value(s) of x does *V* attain its absolute minimum?
(d) For which value(s) of x is $V'(x) = 0$?
(e) For which value(s) of x is $V'(x)$ defined?
(f) For which interval(s) is the graph of *V* concave up?
(g) For which interval(s) is the graph of *V* concave down?
(h) For which value(s) of x does the graph of *V* have an inflection point?

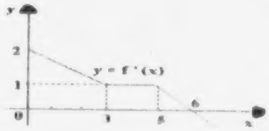
2. The graph at the right is the graph of a function $y = f(x)$. Sketch what the derivative looks like. Give the reason(s) for your answer.



3. Line L is a tangent to the graph of $y = f(x)$ at the point (5, 3).
a) Find the value of $f(x)$ at x = 5.
b) Find the derivative of $f(x)$ at x = 5.
c) What is the value of the function $f(x)$ at x=5.08?
(Be as accurate as possible)



4. The graph shows the derivative function $y = f'(x)$ of a function $y = f(x)$ defined for $0 \leq x \leq 8$.
(a) Find all intervals on which *f* is decreasing.
(b) Find all relative maxima and minima of *f*.
(c) Find all intervals on which *f* is concave down.
(d) Sketch the graph of *f*.

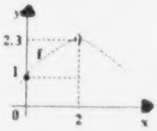


5. Let *f* be a function given by $f(x) = \begin{cases} -x^2 + 4x + 3 & if\ x \leq -1 \\ 2x^3 & if\ x \rangle -1 \end{cases}$

Use the difference quotient to find the slope of the line tangent to the graph of f at
(a) x = 3 (b) x = -1

6. In each of the following situations find the indicated (i) limits; (ii) functional values; and (iii) continuity. If not possible, explain why not.

(a)

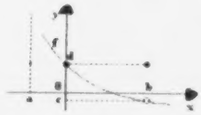


$\lim_{x \to 0} f(x) = ?$ $\lim_{x \to 2} f(x) = ?$

$f(0) = ?$ $f(2) = ?$

Is it continues at x = 0 and x = 2?

(b)

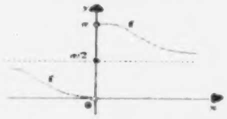


$\lim_{x \to a} f(x) = ?$ $\lim_{x \to b} f(x) = ?$

$f(a) = ?$ $f(b) = ?$

Is it continues at x = a and x = b?

(c)



$\lim_{x \to 0^+} f(x) = ?$ $\lim_{x \to 0^-} f(x) = ?$

$f(0) = ?$

Is it continues at x = 0?

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Supporting Change through a Mathematics Team Forum for Teachers' Professional Development

Paola Valero and Kristine Jess
Royal Danish School of Educational Studies – Copenhagen, Denmark

This paper reports on a first loop of a long-term, action-research inspired, project carried out in a Danish school[1]. The project intends to explore appropriate strategies for professional development in schools, which can support the current reform in mathematics education, through the creation of a permanent mathematics team forum for professional development among the mathematics teachers in the school. The results until now suggest that professional development, if approached as the activity of a community of practice, can build a base for mathematics teaching improvement. Nevertheless, more theoretical and practical work is needed in trying to clarify the meaning of such a community in the case of mathematics teachers in a school context.

Introduction

Current reform proposals in mathematics education in many countries of the world require a strong shift in teachers' practices. The Danish Curricular Guidelines in Mathematics (Undervisningsministeriet, 1995) state general principles about the way teachers have to promote the learning of mathematics in the basic primary and secondary school. The implementation of those principles supposes that teachers participate actively as curriculum designers, something that has not always made part of their practice. This situation of "forced autonomy"(Skott, 2000) makes reforms attempts vulnerable and critical because teachers do not know well how to cope with the new requirements and, therefore, a change does not come as smoothly as expected.

In this paper we report a first loop of an action-research-inspired project that had two main aims. Firstly, it intended to implement a professional development strategy in one school in Denmark. And secondly, it had the purpose of studying the influence of such a strategy on the opening of a mathematics team forum for teachers' professional development in the school.

Theoretical Framework

A recent trend in research on mathematics teachers' professional development conceptualizes school change, mathematics teachers and mathematics teacher education from a systemic, institutional approach (Krainer, 1999; Perry et al., 1998). This trend claims that teachers' practices are strongly connected to and influenced by

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[1] This project has been funded by the Copenhagen In-service Education Department of the Royal Danish School of Educational Studies (RDSES), and has been developed in cooperation with the Center for Learning Mathematics, an inter-institutional research center in mathematics education in Aalborg University, Roskilde University and the RDSES.

the network of practices concerning the teaching and learning of mathematics inside the institutional organization of the school. Therefore, it is necessary to consider, for example, the professional culture of the group of mathematics teachers, and the support structures offered by the administrators for the improvement of that professional culture (Perry et al., 1998).

We would like to emphasize three concepts, which are central to this perspective. *Mathematics teachers' professionalism* refers to the character and nature of the task of teaching mathematics in school (Romberg, 1988). Being professional implies: a) mastering a set of specialized knowledge (i.e., about mathematics, its didactics, the students' learning); b) being aware of the different dimensions of the profession (i.e., its social function, the images of school mathematics associated to one's practice, the consequences of one's actions, the own process of learning to be professional, the recognition of collegiality); and c) acting in agreement to the two above (i.e., making informed and responsible decisions, participating deliberately in a community of practice, seeking transformation; communicating and thinking together with peers).

Professional development is a collective, continuos learning process among the mathematics teachers (Marcelo, 1989). It is oriented towards the improvement and transformation of the practices of teaching and learning of mathematics in the school. And it is enacted through deliberation and coflection, which refer to the social ways of communication and thinking of a collectivity engaged in the learning and teaching of mathematics (Skovsmose & Valero, in press).

Finally, the *fora for mathematics teachers' professional development* refers to the spaces and ways in which teachers can grow in their everyday endeavor. These *fora* are: a) The classroom forum, which designates the spaces of interaction between teacher and students that allow teachers to learn from practice (Steinbring, 1998); b) the external forum, which refers to different in-service, graduate or continuous education taken outside the school (Zaslavsky & Leikin, 1999). These two fora have been considered the main spaces for professional qualification. But if an institutional, systemic approach is adopted, there is a need of a third forum, which is: c) the mathematics team forum, which includes all those opportunities of collegiate work among teachers inside the school during their working time, and which are central to the constitution of a strong professional "community of practice" (Lave, 1996) among the mathematics teachers.

Having all these ideas as a background, we were prepared to design and put into practice a professional development strategy for mathematics teachers and school leaders that could offer some support to the reform process happening in a school. We were also ready to study the effect of that strategy on the possibilities for the creation of a mathematics team forum for professional development in the school.

Description and Methodology

The strategy for professional development took place between June 1998 and May

1999. It involved three pairs of mathematics teachers[2], the pair of school leaders[3] (the headmaster and the vice-headmaster), and us, the pair of "teacher-educators". One of us had previous contacts with the school and we knew about the interest of the headmaster in promoting development in her school. She supported the initiative and negotiated with the local authorities the time that teachers should dedicate to the project. They were given 36 hours. After a first informative meeting for which mathematics teachers in the school had received a written description of the project, these people decided to join voluntarily.

The strategy had the following characteristics. It was not a course and it was clear for all participants that the scheme of expert information-delivery was broken. Instead, all people were expected to have an active participation and to work hard during the whole experience. It considered the school as a basis and, therefore, the teacher-educators mobilized to the school to carry out all the activities there. It was based on the participants' practice because it built upon teachers' knowledge and worries, and not on a syllabus predetermined by the teacher-educators. It was collective and the social interaction was emphasized, so everybody had to work in pairs and, at the same time, participate in the activities of the whole group. It was transformative since it clearly intended to improve a particular aspect of the participants' normal practices. Finally, it adopted a view of learning as a continuous deliberative and coflective process on actual practices.

The whole way of working was heavily inspired by action-research (Kemmis, 1993). Hence, the central task for all participants was to carry out a small inquiry about each pairs' practices. This meant that the administrators had to tackle an aspect of their role as leaders in connection to the mathematics teachers. Each pair of teachers had to inquire about an aspect of their teaching, in one specific mathematical topic, that they found problematic and they wanted to improve according to the proposal of the Curricular Guidelines. And we also had to study our own practice as teacher educators.

The project was organized in six different stages: 1) Information stage, where the teacher-educators contacted the school. 2) Problem definition stage, where teachers and administrators identified a problem or question of their practice that they wanted to tackle. 3) Planning stage, where they had to design a plan to approach the problem. 4) Implementation and observation stage, where they had to bring into practice the plan, and to observe its execution. 5) Evaluation stage, where they had to contrast their initial plan and expectations with the actual implementation, and formulate the results of the experience in connection to the problem previously defined. This included an evaluation meeting of the whole project. 6) Communication stage, where all participants had to write a short report about their own inquiries and presented them in a session with other teachers and guests from outside the school.

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[2] Pairs were formed based on the criterion that both teachers were teaching in the same grade.

[3] The reason to involve the leaders in the project stems from the theoretical assumption of the systemic, institutional approach to school change.

In each of these stages, there were three kinds of activities: 1) Common meetings, where the whole group went through different kinds of activities connected to the main task of the stage. 2) Advisory meetings, where one pair of teachers and the teacher-educators met to discuss in detail the teachers' mini-projects. 3) Pair work, where each pair advanced with the work needed for each stage of the process.

During the whole process, we followed a "discipline of noticing" (Mason, 1997) that allowed us to collect information about our planning and our implementation. We engaged in a constant discussion about our activities and registered them in the form of observation notes, protocols, audio-tapes of some general meetings and some individual meetings, documents handled by the teachers, the school leaders and even our own materials as teacher educators.

Results

In general, participants got involved and worked actively during the whole strategy. All pairs of teachers completed a small inquiry in their classrooms, aiming at understanding more the teaching of a given topic, and at improving their ways of teaching. J. Dyhring and P. Sørensen (1999), in 4th grade, tackled the teaching of isometric drawing and designed a teaching sequence that could allow them to make evident for the students the interplay between one-, two- and tri-dimensions in reality and in graphic representations of those real objects. B. Christensen and H. Jensen (1999), in 6th grade, decided to work with the notion of area, since it is a central concept in all school mathematics and it poses many problems of understanding to students. They designed a teaching sequence that could allow them, on the one hand, to give a more active participation to students in the class activities, and on the other hand, to provide meaning to the notion of area. K. Fink and N. Damløv (1999), in 7th grade, worked with the introduction of linear equations in a more concrete and practical way. L. Spang-Thomsen and S. Fog (1999) started a middle-term work with the group of mathematics teachers, which intended to create a discussion about the theoretical meaning and practical implications of the new Curricular Guidelines in school mathematics.

The implementation process of the whole strategy can be described *a posteriori*, as a chain of events with some critical episodes that determined a change in the interaction and flow of the activities in the project. In what follows, we will provide examples of how, in those different moments, the central ideas of professional development mentioned before came into practice.

A first moment in the process could be called the *entrance moment*. It goes from the first information meeting to the first individual meeting. This process was characterized by the setting of the "educational contract" among all the participants. There was a clear tension between teachers who tried to grasp the words of the teacher-educators, and the latter, who tried to engage the former in the project. One teacher said:

> "We could have been guided a little more from the beginning [...] We didn't exactly know what we were supposed to do, but later on we found out that you knew that very

> precisely […] And you did tell us that the starting phase it was absolutely the worst one." (Jess & Valero, 1999, p. 11)

Despite the tension, all participants got involved and little by little created a comfortable work environment. This moment ended with a critical episode during the second individual meeting. This critical episode was the introduction of concept maps as a tool to organize different thoughts around the plan of action.

A second moment, that could be called the *engagement moment*, goes from the first individual meeting to the report writing. This period was characterized by an increase in the sense making that all participants did of the activities they performed. During the second individual meeting a change in the flow of the interaction in the group was felt. The pair of teacher-educators and each pair of teachers and administrators had the opportunity to go deeper into figuring out how to plan the teaching sequence that they had to implement in their small projects. A teacher expressed:

> "The change in our comprehension happened around the individual meetings, at that time one began feeling high […] and began to feel that something was moving […] personally as well as professionally." (Jess & Valero, 1999, p. 12)

Up to this moment the sense of collective work had not been felt clearly. It arose between the pairs of teachers and the teacher-educators when time was dedicated to think, all together, in the plan of action. The use of concept maps, as a tool to organize those thoughts and come to a more detailed plan of action, was central in creating the need for an active cooperation of all. All participants began realizing that professional development does not happen in individual isolation, but that it is a collective enterprise where diverse institutional actors have a role to play. Administrators and teachers supported each other's activities. In this sense, all started to see possibilities for strengthening a "community of practice" among them. As a teacher noticed:

> "Professionally and collaboratively I have benefited a lot. This has given me so much and has led to that we have talked together about creating something like a collaborative work with the teachers from the same grade, and make teaching materials and other things that all of us can use, instead of closing the doors of our classrooms and making our own individual teaching sequences. […] It could be appropriate to have 5 to 6 main topics around which we [the math teachers] could co-operate during a school year. [It has] been exciting because even if we have been teaching for 30 years, it is only until now that we had a chance of working really close. [We are] two experienced teachers [with] two different ways of working […] and even then we were able to weave us into each other […] That, I think, has been fruitful!" (Jess & Valero, 1999, p. 13-14)

Transformative action was also a central aspect of the whole interaction. The aim of professional development is improving the current conditions and practices of mathematics teachers and administrators. Therefore, they defined projects that clearly allowed them to improve their practices. This sense of transformation was specially lived during the planning and implementation stages of the project, where there was a detailed discussion on possible courses of action. As an administrator said:

> "I think that here we have got an opportunity to see […] how one can get such things to

> work [...] and therefore we have learned something on quite another level, on a real management level, how one can do things like this [...] indeed, we have to establish [...] corresponding professional development in as many subjects as we possibly can." (Jess & Valero, 1999, p. 14)

A third moment in the chain of activities can be called the *summing-up moment*. This period began at the start of the communication stage, where participants had to engage in the report writing process. In the previous engagement moment there was a good mood of work and a nice flow of activities. This flow was altered by a critical episode, which was the setting of the report writing task. From this moment on, new tensions arose. Those had to do with the difficulties in summing up the whole experience, evaluate it a posteriori and write about it. In this moment many of the abilities of the "inquirer" were needed and for many of the participants those have not been experienced before. Although this period posed challenges, it ended successfully with the publication of a booklet (Jess & Valero (Eds.), 1999) that gathered reports from all the projects, and the presentation of them in an open meeting in the school.

The tensions of this moment were overcome because both collective and deliberative interaction arose in the group. As a base for this interaction, the two teacher-educators have deliberately tried to create a power-balanced relationship among them and the teachers and administrators. This was achieved through sharing the responsibility for the project and its functioning –it was not only the responsibility of the teacher-educators that all the activities worked, but also they greatly depended on the commitment of teachers and administrators to their development. We also set a work environment where we all could meet as equal colleagues who have a different expertise, and whose knowledge is valued. As one teacher expressed:

> "You are very good at catching things. One is almost left with the feeling that oneself is a genius [...] Actually it is true; one just says something and then you start working on it. One thinks: My goodness, how smart was what I just said!" (Jess & Valero, 1999, p. 15)

We must also mention that, despite the teacher-educators' efforts to create a power balance –which we consider was positively achieved in terms of giving space to everybody's knowledge and participation– the teachers perceived this issue in a different way. One of the teachers, concerning this point said:

> "You are a bit above [...] That about being equal is only because you have reminded us about it sometimes [...] You have been those who should come with the things. You have the competencies within the different areas, the four pairs of us [...] we have been more equal." (Jess & Valero, 1999, p. 11-12)

We also dedicated time to discussing about topics of relevance for all the participants, and we all made explicit our assumptions, likes and dislikes. A short evaluation period after each session helped setting these conditions.

Concerning teachers' communication, the core of the interaction among all the members of the projects was the engagement in a professional problem-solving process where we analyzed possible courses of action and decided on the most suitable one. Although this kind of activity was present previously during the planning stage, here it played a special role in choosing what and how to give an account of the experience. Connected to this, coflection was especially present because we engaged together in a permanent questioning and criticizing of our choices and actions. As a result we felt that we were learning together by our interaction with the other colleagues.

Conclusions and Discussion

Coming back to one of the aims of this project which was to analyze the way our teachers' and school leaders' professional development strategy had a possible impact on the creation of a mathematics team forum in the school, we could say that the strategy set a basis for it. We found evidence that confirms that the participants empowered their individual and collective knowledge, awareness and capacities of action. Nevertheless, we can not claim that the professional development strategy actually built a strong professional, mathematics teachers' community of practice. Making such an assertion would require longer work with the teachers and a more detailed *a posteriori* inquiry on teachers' and administrators' practices in the school, which we have not made in this first loop of our long-term project.

To conclude we would like to discuss an issue connected to the concepts of professionalization in the context of a mathematics team forum in the school. Many research projects on mathematics teacher education have presented different models to understand teachers' learning, and thus, different meanings of professionalization (see for example "Research Forum: Becoming a mathematics teacher-educater" in PME 23 Proceedings). Most of these models recognize the importance of the cooperative dimension of teachers' work in becoming professional. Nevertheless, we find that such a dimension is seen as complimentary to the individual one, which is dominant. This means, that teachers' professionalization is normally seen as an individual improvement process that, of course, can be strengthened by the social interaction of teachers with colleagues. But what happens if we conceive professional development as a social process in nature –as we suggested in our conceptual framework?

We explicitly adopted an approach that prioritizes the social dimension of in-service teachers' practices and learning. In terms of the strategy we implemented, this approach was clearly noticed in the principles that guided our role as teacher educators, in the tasks we proposed and in our interaction with the teachers. Even the strong sense of cooperation among the two of us was always present and allowed us to exploit teachers' and leaders' interests and worries as a base for our common learning interactions. In terms of the influence of such a strategy on the creation of a mathematics team forum in the school, we have seen that emphasizing that social dimension could be an appropriate way of dealing with professionalization. If the latter intends to be seriously connected to the existence of a "community of practice" among the mathematics teachers in the school, then we need to develop a deeper understanding of the social dynamics of that community (Matos, 1999). Through the entrance, engagement and summing up moments previously described we could identify how a collective sense, a transformative purpose, a deliberative communication and collective thinking ways were emerging in teachers' and leaders' work. Still, there is a need to develop further, in theory and in practice, the meaning of all these concepts if we want to support the reform process in a significant way.

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STUDENT TEACHERS' CONCEPT IMAGES OF ALGEBRAIC EXPRESSIONS

Nelis Vermeulen Cape Technikon Cape Town

Pre-service first-year mathematics education students often enter training programmes with established concept images regarding algebraic expressions that are not associated with successful mathematics teaching practice. This paper describes the suggested current teaching approach towards early algebra, in particular algebraic expressions, in South African schools, the orientation of student teachers with inappropriate concept images of algebraic expressions towards this teaching approach, and how difficult it appears to change these students' concept images to match the suggested teaching approach. The results, obtained in writing and through interviews, emphasize concept images' resistance to change and provide evidence of contradictory concept images present within the same learner.

1. INTRODUCTION

There are currently at least three approaches towards teaching early algebra in South Africa:

a) The traditional approach with an overemphasis on **manipulative skills**

b) An approach which allows far more for **conceptual development** before manipulative skills are introduced

c) A **problem-centred** approach which has not been implemented widely yet.

The syllabus in use since 1996 suggests approach b).

The strategy generally followed by textbooks supporting approach b) regarding algebraic expressions can be described by the following phases:

- Algebraic expressions are constructed from everyday life situations and generalizing activities, developing conceptual understanding of the meaning of letter symbols as placeholders for numbers, the meaning of algebraic expressions as formulas or computational procedures, and of the function concept through the substitution of several values in the place of the variable and the resulting variation of values.

- Development of the concept of equivalent algebraic expressions, usually through substitution of various number values into expressions such as $2x + 5x$, $10x - 3x$, $7x$, etc. or $(3x + 5) + (2x + 3)$ and $5x + 8$, and the concept of an identity, e.g. the meaning of the statements $2x + 5x = 7x$ and $(3x + 5) + (2x + 3) = 5x + 8$. From this should follow an awareness that some equivalent expressions are more convenient than others for a particular purpose, e.g. it is much easier to evaluate $5x + 8$ than $(3x + 5) + (2x + 3)$.

- Clarifying why certain expressions are equivalent to others, that is, what are the principles that ensure equivalence. These are the properties of operations, namely the commutative and associative properties for addition and multiplication and the distributive property of multiplication over addition. "Algebraic manipulation is based on exactly the same general properties of numbers than arithmetical manipulation....this then is the true sense in which manipulative algebra is generalized arithmetic" (Human, 1988:9).

Manipulative skills are only developed at this stage, with the emphasis on simplification as the process of replacing one expression by another equivalent expression in the most precise and economical form (Booth, 1986).

This approach corresponds to a number of well-known views regarding the teaching and learning of early algebra, e.g. Booth (1983: 75): "Algebra is to be regarded, at least in its elementary stage, as the representation in general form of the operations and structures of arithmetic...Research in algebra should look beyond acquisition of algebraic skills to a consideration of the conceptual framework within which those skills might be constructed"; Skemp's well–known principle of relational vs instrumental understanding, and his notion of surface structures and deep structures (Skemp, 1982) ; Booth (1986: 2): "One of the most important functions of algebra is to permit the concise representation of general relationships and procedures... Closely related to the appreciation of the meaning of a generalized statement is, of course, the view of the letters contained in that statement as 'variables'. Indeed, one could say that until a student does appreciate the use of letters as variables, or at least as generalized number, then algebra can have little real meaning"; and Linchevski (1995): "...an understanding of variables and algebraic expressions might be built through generalizing activities with number patterns...".

Although it can be argued that learning mathematics does not occur in a linear fashion, and that ..."a curriculum carefully built in this way can cause serious difficulties in learning" (Tall, 1989:37), the phases of the suggested teaching strategy can be considered as outcomes which need to be achieved in order to form appropriate concept images of algebraic expressions, rather than prescribing a particular sequence.

2. THEORETICAL FRAMEWORK

This research is carried out from a constructivist perspective, in which it is accepted that mathematical knowledge cannot be transferred ready-made form one person to another, but that each individual continually constructs his or her personal knowledge, based on his or her own experiences, and in terms of his or her own, existing knowledge base. The view of learning as an active construction process implies that learners build on and modify their existing concept images, that is, through the processes widely known as assimilation and accommodation.

Sfard and Linchevski (1994) point out that eventually all mathematical conceptions are endowed with a "process-object" duality. For example, an algebraic expression such as $3(x + 5) + 1$ may be interpreted in several different ways, for example

- it is a concise description of a *computational process*, a sequence of instructions: add 5 to the number at hand, multiply the result by 3 and add 1.
- it represents a certain *number*. It is a product of a computation rather than the computation itself. Even if this product cannot be specified at the moment because the number x is presently unknown, it is still a number and the whole expression should be expected to behave like one.
- it may be seen as a *function* – a mapping which translates every number x into another. The expression does not represent any fixed (even if it is unknown) value. Rather it reflects a change; it is a function of two variables.
- it may be taken at its face value, as a mere *string of symbols* which represents nothing. It is an algebraic object in itself. Although semantically empty, the expression may still be manipulated and combined with other expressions according to certain well-defined rules.

The *plurality* of perspectives may seem confusing, but it is actually necessary that the learner should develop *all* these different meanings, because they are used in different contexts.

According to Tall & Vinner (1981) a person's concept image consists of all the cognitive structure in that individual's mind that is associated with a given concept. Thus, a concept image includes all the mental pictures and associated properties and processes he/she associates with the concept (Vinner, 1983). It is built up over time through experiences of all kinds, changing and maturing as the individual meets new stimuli. As the concept image develops, it need not be coherent at all times.

Considering teaching approaches and textbooks utilized, linked to numerous observations, it seems reasonable to argue that the **mental picture** regarding algebraic expressions of a learner who is the product of approach a) could be: algebraic expressions are collections of terms, consisting of numbers and symbols. This would correspond to Sfard and Linchevski's perspective of "a mere string of symbols". On the other hand, a learner instructed through approach b) could have the following mental picture: algebraic expressions as a short-hand way of writing sets of computational procedures or instructions, where the symbols represent number values. This would largely correspond to the first three of Sfard and Linchevski's perspectives.

Properties and processes associated with algebraic expressions in the mind of a learner from approach a) could be: (some) expressions can be simplified, brackets should be removed, and like terms should be added and subtracted. In the mind of this learner, these procedural properties are often based on rules prescribed by some authority such as the teacher, and are usually completely severed from the properties of operations of real numbers. There may also exist an urge to "do something" with the expression, i.e. to simplify, remove brackets, add or subtract terms, etc. whether it is appropriate or not. (Refer to some of the excerpts in 5. Results). For the learner from approach b) properties and processes associated with the concept of algebraic expressions can be: equivalence, ensured by the application of properties of operations of real numbers, which are the same as those utilized in arithmetical calculations, thus enabling the learner to comprehend the relationship between arithmetical calculations and algebraic manipulations. Indeed, Kieran (1989) emphasizes that an important part of learners' difficulty in learning the basic ideas of early algebra is their difficulty in recognizing and using systemic structure, i.e. that algebraic expressions can have equivalent forms, obtained by applying the properties of operations.

The different concept images described above can (at least in part) be ascribed to the role of the formal **concept definition** (words used to specify that concept). Text books used in approach a) define an algebraic expression very early. Examples are: "An algebraic expression consists of numbers and symbols, joined by + and – signs to form separate terms" or "6,5 + 25 – 7,84 is an arithmetical expression consisting of three numbers (or terms). If the numbers in this expression are replaced with symbols, the expression becomes an algebraic expression. For example, a + b – c is an algebraic expression of three terms". The concept is therefore defined before the learner had an opportunity to form a concept image. In approach b) the formation of a concept image is encouraged before introducing a concept definition. Approach a) corresponds with a statement by Vinner (1983), according to whom it seems as if many teachers at the secondary level expect learners to form concepts in a one way fashion, as shown in the sketch below, namely that "the concept image is formed by means of the concept definition and under its control. We, however, consider this as wishful thinking".

Concept definition → Concept image

Tall (1990) states: "...we cannot improve matters by simply giving better concept definitions...because what might be an appropriate foundation for a logical mathematical development may not be an appropriate starting point for a cognitive development".

At different times, seemingly conflicting concept images can be evoked. This can happen simultaneously, and can lead to a sense of conflict or confusion (Tall and Vinner, 1981).

3. CONTEXT

As part of their pre-service course at our institution, prospective mathematics teachers study a module called Teaching Early Algebra. The primary objective of this module is to prepare these student teachers to teach algebra in grade 8 utilizing approach b). A secondary, but equally important, objective is to enable students to develop all the different meanings of algebraic concepts, including algebraic expressions, as proposed by Sfard and Linchevski (See 2. Theoretical Framework).

Most of these students are from the so-called historically disadvantaged group in South Africa where mathematics education at school generally follows approach a) above (an overemphasis on manipulative skill), and where they receive tuition in their second language.

Part of the challenge of this **ongoing** research project is to enable students to change their concept images from those associated with approach a) to more appropriate ones, that is, those associated with approach b). However, there is in general a strong resistance to any change in existing concept images. Learners often firmly believe in their own constructed ideas and are not readily prepared to introduce major changes. Although learners may experience that their previously constructed knowledge is inefficient, it is not self-evident that their concept images will be restructured to accommodate the 'new' knowledge (Bezuidenhout, 1998).

4. METHODOLOGY

During 1997 the module Teaching Early Algebra was presented as a series of lectures. The nine students' knowledge and understanding of this module were assessed by them answering the question: "Describe a strategy to teach early algebra in the secondary school. Explain the relevance of each of the phases of this strategy and give examples of activities to support this strategy". During 1998 (seven students) the method of instruction and the assessment were similar to that of 1997. During student assessment of this module in 1997 and 1998 students demonstrated a lack of comprehension of this approach towards teaching early algebra. Given the nature of their mathematics education, I hypothesized that these students brought with them concept images which are quite different from the ones needed to teach early algebra as suggested by this module, that an incongruence existed between their concept images and the ones required for teaching early algebra based on approach b), and that the method of instruction I followed teaching this module did not succeed in changing or improving their concept images. At this point I decided to conduct interviews with these students to investigate their concept images regarding algebraic expressions. These interviews to a large extent supported this hypothesis. I therefore restructured the module for 1999 to enable students to experience this approach to early algebra in a hands-on fashion. During 1999 students (eight of them) were first given activities in which to engage. These activities were similar to those that form part of the suggested teaching strategy (approach b)). Once they have completed the whole range of activities, the framework (phases) of the suggested teaching strategy was discussed and clarified. Assessment was similar to that of 1998, i.e. a question to be answered, followed by interviews. The purpose of the latter once again was to determine the nature of their concept images.

The interviews focused on students' concept images of **algebraic expressions**. This includes students' understanding of: the meaning of letter symbols and of algebraic expressions, the notion of equivalence, the role of properties of operations to ensure equivalence, and the similarities between algebraic manipulation and arithmetical calculations based upon properties of operations.

5. RESULTS

5.1 Written assessments in 1997 and 1998: Many students' answers revealed that they had little appreciation for the relevance of each phase in developing conceptual understanding of algebraic expressions, that certain phases were not even mentioned in their answers, and that examples of activities were either not given or inappropriate. This prompted the interviews to determine their concept images.

5.2 Written assessments in 1999: In general students' answers were an improvement over those of the previous years, but they still did not demonstrate a sufficient understanding of this approach.

5.3 Excerpts from 1999 interviews:

Students' understanding of the *meaning* of letter symbols and algebraic expressions:

I: *When I write down $3x + 5$, what does it mean to you?*

M: *Unlike terms. You cannot add unlike terms.*

I: *OK, unlike terms. Anything else that $3x + 5$ means to you?*

M: *(Silence) No.*

I: *Can you give it a name?*

M: *Series of numbers.*

(This student apparently realizes that an algebraic expression can be viewed as a computational procedure, i.e. a series of calculations, but may lack the language ability to express this).

I: *OK, a series of numbers. How will you explain that to me?*

M: *You cannot add, you cannot divide, it just stays like this.*

(He is very aware that $3x + 5$ is the simplest form for this expression).

I: *If I write for you something like $3x + 5$, what does it mean to you?*

V: *An expression*

I: *Why?*

V: *Because there are known numbers and unknown numbers it is something that I have to calculate.*

(He appears to realize that an algebraic expression can be viewed as a computational procedure, rather than merely a string of symbols).

I: *What is the unknown?*

V: *x.*

I: *How are you going to calculate?*

V: *You calculate the expression how can I say you calculate it to the instruction given to you. If they say you are going to solve for x. If they say you must find the way they instruct you I will calculate it if you give me an instruction. I can't calculate it if you don't give me an instruction.*

(This student demonstrates the urge to do something with the expression).

I: *OK, so $3x + 5$ is an expression, but it does not mean anything to you unless there is an instruction.*

V: *Yes, an instruction.*

I: *Now, what type of instructions can there be?*

V: *Hmmm can ask me to draw a graph.*

I: *Can you draw a graph of $3x + 5$?*

V: *No.*

I: *Why not?*

V: *It's not an equation.*

I: *What other types of instructions?*

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V: *(Silence)*
I: *Solve?*
V: *Yes, they can say solve for x.*
I: *OK, what are you going to do ...*
V: *... or they can say find an equivalent expression.*
(In spite of the (perceived) initial demonstration of his awareness of an algebraic expression as a computational procedure, the urge to do something with an expression discloses a very confused concept image).

I: *If I write there for you 3x + 5, what does it mean to you?*
D: *... ... ... you need to find x ... you need to say 3x + 5 is equal to zero.*
I: *Hmm...*
D: *Then you take the 5 to the other side and ...*
(The urge to do something; even change it into an equation).
I: *So you mean that 3x + 5 is an equation and it is equal to zero?*
D: *Yes.*
I: *Is there any other view that you have on 3x + 5, except that you can solve it ... find the value of x?*
D: *... ... ... No.*
I: *Right, let me ask you: that x there, what does that x mean to you?*
D: *... ... x stands for any number.*
I: *Can x also stand for something else, except a number? Can it stand for people?*
D: *Yes ... ...*
I: *Can it stand for apples?*
D: *... ... ... numbers.*
I: *Only numbers? Not people? Not apples?*
D: *(Silence)*
I: *Can I say 3x means three apples?*
D: *... ... Yes, we can say that.*
I: *So you say 3x can mean three apples?*
D: *... ... Yes.*
(Evidence of two conflicting concept images. From me she learned that letter symbols represents numbers, and that was her first reaction to my question above. However, her concept image from school is so strong and resistant to change that she still entertains it. Her hesitation in answering may be indicative of the cognitive conflict).

6. CONCLUSION

From the interviews it became clear that in general students' concept images were still not reflecting a proper understanding of notions surrounding algebraic expressions. Evidence of concept images based on the type of teaching they received (approach a) was abundant, as was the simultaneous presence of old concept images and newly formed, but far from stable and comprehensive, ones, which gave rise to cognitive conflict.

The instructional approach followed with the 1999 students obviously did not produce the results hoped for. Perhaps the students of 2000 should be exposed to a problem-centred approach, followed by a discussion of the phases. Perhaps more time is needed for the changing of their concept images.

It appears as if Lee (1985) may have the last word on this: "Because a large percentage of students are entering teacher education without the necessary math skills to complete the program, the onus of responsibility of college instructors points not to advancing mathematical expertise, but rather to the remediation of existing skills."

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Spreadsheet mathematics in College and in the workplace: a mediating instrument?

G.D. Wake, J.S. Williams & J. Haighton

University of Manchester, U.K.

This paper highlights some aspects of case studies of College students' understanding of workplace practices with spreadsheets in the police service and elsewhere. It seems that there are aspects of spreadsheet mathematics that 'transfer' quite readily from the College to the workplace contexts. These involve working within the spreadsheet models. On the other hand, College students had more difficulty when appreciation of the meaning of the model was involved in making sense of the activities. We identify the potential of a general modelling strategy in helping both students and workers to develop understanding of the models they use.

Introduction and background

This paper reports a case study investigating how College students use mathematics to make sense of workplace practice. A number of workers in different workplaces identified activity where they analysed various aspects of workplace performance as a possible area of interest for our research. Such activity often involves their use of computer spreadsheets to analyse and display data both numerically and graphically. This report describes and analyses such activity.

A substantial and growing body of research has sought to understand how individuals use mathematics in 'everyday' or workplace practice (e.g. Lave, 1988; Lave & Wenger, 1991; Chaiklin & Lave, 1993). Proponents of situated learning or cognition suggest that individuals develop their understanding and use of mathematics within their situated activity or practice; as they move from one situation to another they have to reconstitute their understanding to take account of the distinctive features of the new situation (van Oers, 1998). Mathematics outside school, such as in workplaces has been contrasted with mathematics in school (e.g. Nunes et al, 1993; Saxe, 1991). Our Leverhulme project attempts to explore the issue of 'transfer' or transformation of knowledge between college and work contexts by introducing the 'voice', in Wertsch's sense (1991), of the college student to dialogue in the workplace.

We use Engestrom's activity system schema (see for example, Engestrom, 1991) to assist in illuminating how mediating instruments, community rules and the division of labour in the different systems of workplace and College act to influence the mathematical experiences of college students and workplace operatives

(Williams et al, under review). However, it is the object of each activity system that predominates and acts to organise the cognitive processes of the individuals involved. Consideration of the different objectives of college and workplace activity systems have allowed us to shed light on why, when cognitive tools, such as graphs, appear the same to external observers, they can prove problematic to the student operating in each of two distinctly different activity systems. We suggest that a graph can be considered to have a 'type' meaning which is the same in all contexts (a visual representation of data) and numerous 'token' meanings in particular contexts. A semiotic analysis of transfer allows us to consider how signs (such as graphs and measures) are carried across contexts but their meanings may be considered to be transformed by chains of signification (e.g. Cobb et al, 1997; Presmeg, 1999). Such analysis has allowed us to understand how problems arise for students when attempting to make sense of workplace practice with their college mathematics.

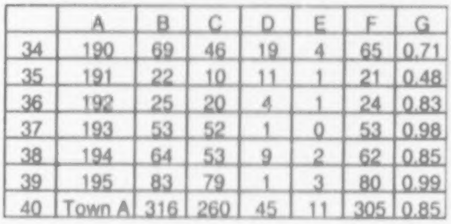
Each case study we develop involves interviews with workers, and their managers, in which we explore the nature of their work. A further series of interviews with teachers and students allows us to familiarise ourselves with the students and their academic background. The vital generative ingredient in the case study involves bringing together the student and the workplace, and the worker, student and mediating researcher to explore the nature of the workplace activity together. This allows us to examine and question closely whether college mathematics can be used to make sense of workplace activity, and also what is the role of the researcher/teacher in mediating between mathematics, student and workplace?

Using spreadsheets to analyse workplace performance

In three separate workplaces we investigated workers making sense of data to determine whether or not performance targets were being met or to assist in decision making. Two of the workplaces investigated involved finance offices; the third workplace involved a police policy and performance unit. This unit monitors a wide range of police activity in terms of the implementation of national and local policy and the performance of police personnel for a 'Division' based on five towns. One important indicator of performance, determined by central government, is the time taken to respond to emergency calls to the police. Such calls should be responded to within ten minutes in urban centres and within 20 minutes in rural areas. Data for the Division is forwarded by a central monitoring unit to an Inspector who analyses this and presents a visual display to other officers in charge of the police stations in each of the towns. The policing within each town is broken down further into 'beats' to which each individual police officer is assigned. Eventually the Inspector's analysis of performance will be interpreted by all personnel in the Division.

Figure 1 shows part of a spreadsheet developed by the Inspector to analyse the data supplied by the central monitoring unit and Figure 2 gives one graphical output he has developed based on this. Figure 3 shows a further graph the Inspector uses to show the performance of the towns over a 12 month period. This work of the Inspector was explored at a meeting at which the Inspector, a Police Constable (responsible for community affairs), two college students and one of the authors as researcher were present.

The two students had both expressed a wish to eventually work in the police force; each had only just reached the minimum standard required in mathematics for entry and neither included mathematics as part of their post-16 programme of study. At the meeting the Inspector described to the group how each month he inputs data into his spreadsheet and how this is analysed to give a numerical and graphical display.



	A	B	C	D	E	F	G
34	190	69	46	19	4	65	0.71
35	191	22	10	11	1	21	0.48
36	192	25	20	4	1	24	0.83
37	193	53	52	1	0	53	0.98
38	194	64	53	9	2	62	0.85
39	195	83	79	1	3	80	0.99
40	Town A	316	260	45	11	305	0.85

Figure 1. Spreadsheet of performance data

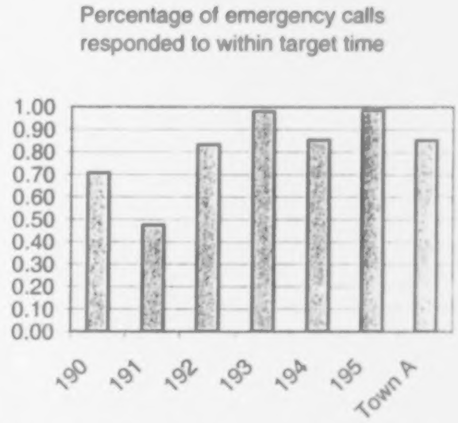


Figure 2. Graphical output of performance data

Inspector: … this is just raw data we get from Police Headquarters, so we get the beat [indicating column A], how many calls were received [indicating column B], right? How many were on time [indicating column C], yes? How many were not on time [indicating column D], and how many were discounted [indicating column E]. … I've set that [indicating cell F34], look, to SUM B34 minus E34, so you've got B34 minus E34. … So the end column then, is SUM C34 – which is that one – divided by F34, which gives you percent, right?

Conversation confirmed that the Inspector only ever considered the formulae he used on the occasion when he first generated the spreadsheet.

The numerical output (see column G) gives the success rate as a decimal number. This eventually caused some problems for the Inspector although he could translate comfortably between decimal number and percentage in conversation, and could articulate the conversion operation. He was, however, unable to translate this thinking into his spreadsheet and had to re-type the figures as percentages into a different part of the spreadsheet to produce the historical graph (Figure 3). The students were equally comfortable with the required conversion and recognised that the graphical display of Figure 2 presents the data in decimal form rather than as a percentage. During our conversation they assisted the Inspector in adapting his spreadsheet to recalculate the values in column G as percentages.

Inspector: How can I get that decimal point moved to there [indicating two places to the right of its current position], basically, isn't it?

Student: Just take out your decimal and just use your ordinary numbers... after the decimal..[referring to formula '=SUM(C34/F34)] ... when you close-bracket-times-a-100 from there

Inspector: Times.... times...

Student: 100.

Inspector: And then if I press Enter, that'll mean.. the...*(laughter at his success)*

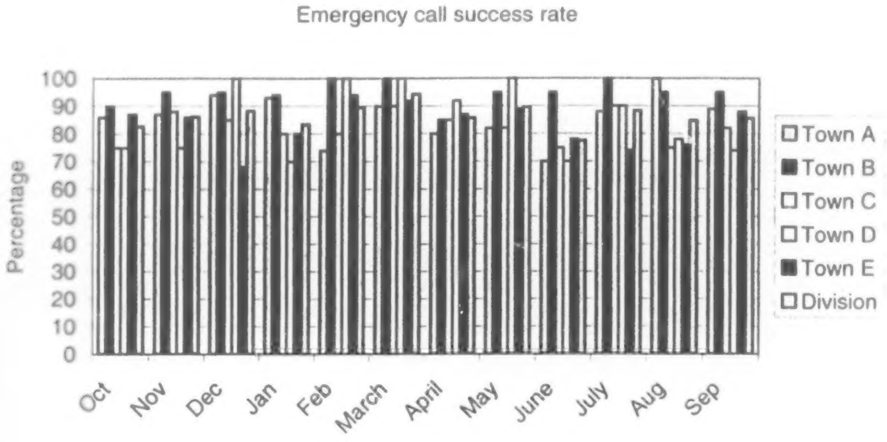


Figure 3. Graphical output of historical data

During exploration of the spreadsheet display it became apparent that the decimal values in column G had been rounded. The Inspector used a button on the spreadsheet toolbar "to take decimals out". He sees this as useful in helping others make sense of the data and is aware that the issue is to some extent one of display.

Inspector: [the spreadsheet] is still working out all these point-whatevers, even though it's not displaying. All that decrease and increase does is alter the visual image on the screen.

One of the students explained to the Inspector the underpinning mathematics, "When it gets past 74.5 it'll go up to 75," going on to explain when values are rounded up and when they are rounded down.

However, not all of the mathematics involved in this case study proved totally unproblematic for the students - measures were used where the police officers and students did not have a sound understanding of the basis on which they were developed. In this case some light was shed on exactly what the measures convey by the researcher using a general strategy of considering extreme values.

The Inspector had found the average for the Division as a whole by averaging the averages for each of the five towns. The researcher questioned the validity of doing this as each town has a different number of beats. There seemed no problem to the police officers and students until the researcher suggested,

> "if you imagine ... one of your divisions has only got one beat in it, and the other one's got a 100 beats in it ...".

All involved in the conversation could then identify a potential problem; the Police Officer asked,

> "Do you think it's unfair because you've got more beats in one area than another ... So somebody's carrying a bigger burden..? ... I know what you're getting at, yes. We have to take the weighting out of it."

The Inspector agreed with the Officer and asked if the spreadsheet had an in-built function to calculate a 'fairer' measure. By this stage of the meeting the students had understood sufficient of the situation to be able to identify the town that was contributing most to the 'unfairness'. Although both students and police had neither the correct mathematical language to talk about the problem nor the technical competence to surmount it, they did have, in this context, in which the problem (of unfairness) was intuitively manifest, an understanding of its nature.

This strategy of considering extreme, or simple, values again proved useful in another case-study in which we investigated similar activity of workers analysing performance data - in a finance office of a medium sized company. In this case a measure "debtor days" is used to give an indication of how long customers are taking to clear their debts. This measure is found using the formula

"debtor days = (outstanding debts / annual turnover)*365".

The worker who calculated this measure each month, although having a sense of what the measure conveys was not able to make sense of how the measure related to the data involved. The researcher and office manager, however, were able to gain an understanding by substituting the simplified values, "annual turnover = 2 million" and "outstanding debts = 1 million" giving "debtor days = 182.5" or half a year. This, therefore, gives an indication of how long customers are taking to clear debts - although it does use actual information it does not relate to real data about how long actual customers take to clear debts.

Analysis and discussion

The case study reported here gives us some hope that mathematical knowledge can in certain instances be 'transferred' from college to workplace with some ease. The most readily transferable skills were those that relate directly to operations within the spreadsheet: i.e. the use of the technical aspects of the tool working within the model as it were, prove easy to transfer from college practice. Students were not only able to apply mathematical ideas to make sense of the spreadsheet work of the police Inspector but were also able to gain some understanding of his interpretation of the analysis. However, as is to be expected, this was not to the same depth as that of the police Inspector who for instance was able to explain to the students why a certain 'beat' based on a motorway had a high success rate and why another 'beat' classified as 'urban' was in fact two urban areas split by a large tract of rural land consequently leading to a low success rate. The Inspector summed up his involvement in making sense of his analysis:

> "But because I'm *living* it and I understand what figures I'm putting in to it, it's interesting. After each one, I'm thinking, 'Crikey, 'he's up, or he's down...".

The Inspector had constructed the spreadsheet and its numerical and graphical output in response to a clear need. This had involved him in finding formulae and methods that work, even though some of his techniques (such as his use of the formula "=SUM(C34/F34)") would not be introduced in mathematics classrooms, to meet his objectives. The students were able, on a number of occasions, to offer alternative solutions that appear to reflect clearly the experience of their formal college mathematics such as their suggested replacement of the use of the spreadsheet formula "=AVERAGE(H6:H10)" with "=SUM(H6:H10)/5".

It appears that the spreadsheet acts as a mediating 'tool' assisting in the semiosic processes that construct a bridge between College and workplace cultures. The spreadsheet is therefore potentially very powerful, in that it may allow students to transform classroom knowledge with relative ease into workplace settings. Although initially designed as software for business applications it has been adopted by other workplaces and indeed education as a tool that allows us to use spreadsheet mathematics to make sense of data. Perhaps it is the adaptation by the educational industry of a workplace tool that gives it this power as a mediating instrument.

On the other hand our case studies have highlighted that problems of 'transfer' still remain when the mediating influence of the spreadsheet is not apparent - when workers and students are required to critically think about, or understand, the measures they calculate, and on occasions how they carry out their calculations. The strategy adopted by the researcher of considering the meaning of extreme or simplified values for the variables in a procedure, rule or formula to help workers and students explore its validity is a *general* strategy which does not depend on knowing much about the situation or activity. It seems that workers and students do not have such general strategies at hand that allow them to investigate the meaning of calculated values; these may need to be introduced and taught, for instance as modelling skills. Future work might usefully explore this as a category in discussions involving students and workers; can the researcher sometimes bridge gaps in understanding by the use of such general strategies? Indeed curriculum developments taking place in parallel with this research in the U.K. have resulted in new qualifications (Wake, 1997) based on general mathematical competences (Williams, Wake & Jervis, 1999) which emphasise the importance and evaluation of mathematical models used in the application of mathematical principles in context. As the resulting curriculum develops we hope to be able to explore more deeply students' understanding and application of such modelling strategies in both classroom and workplace settings.

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Visualisation and the development of early understanding in algebra

Elizabeth Warren,

Australian Catholic University

This paper explores the role of visualisation in the algebraic domain, and in particular how the ability to visualise assists students in reaching generalisations from visual patterns and tables of values. Two written tests were administered to 379 students and from the results students were selected for semi-structured interviews. The results of the written tests and the semi-structured interviews indicated that both spatial visualisation and pattern completion reasoning processes help students reach generalisations and link representations.

Didactic cut – the transition from arithmetic to algebra

Algebra consists of two components, representing numbers and quantities by literal symbols and calculating with these symbols. This focus is on a letter-symbolic form. By contrast, algebraic thinking refers to a broader range of representations. Students are engaged in algebraic thinking when they represent the relationships between quantitative situations. This representation could include the letter-symbolic form but could also include graphs, spreadsheets, geometric patterns and natural language. Beginning algebra students are required to move from arithmetic thinking to algebraic thinking. This transition is referred to as pre-algebraic thinking, the didactic cut between arithmetic and algebra.

Boutlon-Lewis, Cooper, Atweh, Pillay, Wilss, and Mutch (1997, 1998), Cooper, Boulton-Lewis, Atweh, Pillay, Wills and Mutch (1997) studies examined students' growing ability with algebraic concepts and processes, namely, the commutative, associative, and distributive laws; inverse operations; order of operations; meaning of equals; and, meaning of the variable. One outcome of this longitudinal study was the proposal of a model that is believed to describe algebraic development. The model is based on the premise that students' difficulties stem from dealing with inappropriate cognitive loads in their early algebraic experiences. While this theory is appealing for sequencing the introduction of algebraic expressions, it fails to recognise the teaching approaches and resources commonly used to aid this transition, and in particular the specific thinking that helps student distil the essence from these approaches.

Common approaches used in this transition

Two approaches for bridging this gap are recommended in the literature. The first involves generalising the patterns in arithmetic and the second involves generalising the patterns found in functional situations such as number patterns, visual patterns, and tables of values. This functional approach is a response to the need to contextualise the initial instruction in algebra, with the focus being on the language of generality. The use of many representations is a means of offering a bridge between the symbolic manipulations in algebra and knowledge of functions (Yerushalmy, Shterenberg & Gilead, 1999). This paper examines the role that visualisation plays in success with the functional approach.

Problems children are having with these approaches

Recent research has identified many difficulties that students experience when using a functional approach to introduce the variable (Redden, 1996; Stacey & MacGregor, 1995; Warren, 1996). The role that language plays in success has received considerable attention. Many students experience difficulties with expressing the pattern symbolically. Kaput (1992) suggested that these difficulties could be due to the fact that the symbolisation schema is a hybrid of arithmetic symbolisation schema and transliteration schema based on natural language. Redden's (1996) results indicated that natural language descriptions of number patterns seem to be a necessary prerequisite for representing the patterns in algebraic notation. It appears that attention to language is important for success. The role visualisation plays in generalising from these visual patterns and linking the representations used with these approaches has received limited attention.

Visualisation can be defined as a kind of reasoning based on the use of mental images (Guttierez, 1996). For this research, visualisation is closely aligned with spatial reasoning. It has been argued that spatial reasoning consists of two distinct components, spatial visualisation and spatial orientation (Tartre, 1990). The former involves the skill of mentally manipulating, rotating, twisting or inverting a pictorially presented stimulus object. By contrast, spatial orientation involves the skill of understanding a spatial representation or comprehending a change that has taken place between two representations. Effective spatial thinkers are better able to construct, visualise, transform, interpret, and classify geometric shapes, patterns, and diagrams.

Patterning in pre-algebra and early algebra involves a process of creating and extending patterns, and from these data, drawing appropriate generalisations. The patterns could be numbers or visuals. The facility with patterning includes the ability to identify, analyse, and describe patterns, draw appropriate generalisations and from these identifying the next step. This is referred to as the pattern completion reasoning process.

The specific aim of this study was to investigate the role spatial visualisation, spatial orientation and the pattern completion reasoning process play in reaching generalisations from visual patterns and tables of data.

Methodology

Instruments

Two instruments were developed for the research, an algebra test and a reasoning process test. The algebra test consisted of three components, namely, generalising from visual patterns, generalising from tables of data, and understanding the concept of a variable. The reasoning process test consisted for five components (logical, analogical, pattern completion, spatial visualisation, and spatial orientation). For this report, only the visual components of this test are considered (pattern completion, spatial visualisation and spatial orientation).

The patterning component of the algebra test involved reaching generalisations from the first four steps of a visual pattern, and the table of data component involved reaching generalisations from partly completed tables of data.

For the reasoning process test, each item chosen for the pattern completion component involved the skill of examining the first four steps in a pattern and from these identifying the next step. For the spatial visualisation component, all items chosen involved the skills of either mentally moving separate pieces to form a particular pictorial representation or mentally folding two dimensional patterns into their associated three dimensional state. By contrast the spatial orientation component required the perceptual perspective of the person viewing the object to change. The items comprised Gestalt completion tasks.

Both written tests were administered to a sample comprising 379 students from one independent coeducational school and one state coeducational school in the metropolitan area. The students' ages ranged between 12 years and 2 months and 15 years and 10 months. Both schools chosen for the study were large metropolitan schools consisting of students from lower-middle socio-economic status, with a variety of ethnic backgrounds represented. From the results of the written tests students were selected for in-depth interviews. This provided insights into the ways students' think, and enabled clarification, extension, and interpretations of the information recorded in the written responses.

Results

Chronbach alpha coefficients were calculated for each component of the reasoning process test. The results (Pattern completion (0.84), Spatial visualisation (0.64) & Spatial orientation (0.60)) indicated that the items used in these components were considered to be adequate measures.

In order to ascertain the interaction between the three visual reasoning processes and the components of the algebra test, a Spearman Rank Correlation was performed. Table 1 summarises the results of this analysis.

Table 1

Spearman Rank Correlation between reasoning processes and the three components of the algebra test.

	Algebra test		
Reasoning process	Patterning	Tables	Variable
Pattern completion	.33*	.27	.23
Spatial visualisation	.34*	.31*	.31*
Spatial orientation	.07	.04	.02

* Correlations of education significance (r>.3).

The pattern completion reasoning process was significantly correlated with the patterning component of the algebra test. Spatial visualisation correlated significantly with all three components of the algebra test. The spatial orientation reasoning process failed to correlate significantly with any components of the algebra test. Success on the patterning component of the algebra test seemed to require both the pattern completion and spatial visualisation reasoning processes, whereas, success on the table of data component of the algebra test only required spatial visualisation.

From the results of the written tests students were chosen for an in-depth interview. This paper reports on four groups of these students, namely, the students at the 100th and 0th percentile of the patterning component of the algebra test and students at the 100th and 0th percentile of the table component of the algebra test. Each group consisted of four students. After these students were selected, the group mean score for the spatial visualisation and pattern completion reasoning processes of the reasoning process test were calculated. Table 2 presents the mean score for each group.

Table 2

Group mean score for the spatial reasoning test and pattern-reasoning test

Algebra test	Groups	Spatial visualisation	Pattern completion
Patterning	100th percentile	8.75	8.75*
	0th percentile	7.25	4.75*
Table of Data	100th percentile	7.75*	6.5
	0th percentile	3.35*	4.25

* significant difference between groups

Mann-Whitney tests were performed to compare the two patterning and two table of data groups. The overall significant level was set at 0.05, and the Bonferroni Inequality (Stevens, 1992) was employed because two analysis were performed for each group. The conservative application of this inequality required the planned Type 1 error for each analysis to be set at the family-wise level divided by the number of analysis (i.e., .05÷2 = .025). The two patterning groups were significantly different for pattern completion reasoning process and the two table of data groups were significantly different for spatial visualisation reasoning process.

Two of the tasks presented in the semi-structured interview were believed to probe how spatial reasoning and the pattern completion reasoning processes helped students reach and link generalisations. Both tasks were also included in the written test. Figure 1 and Figure 2 present the two tasks.

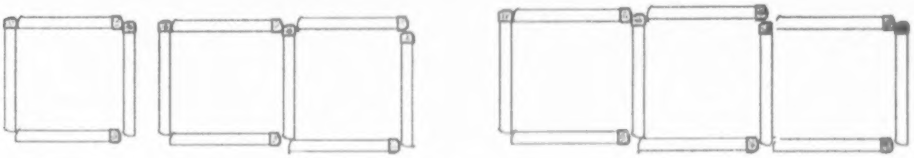

Figure 1 The Patterning task used in the semi-structured interview.

Students were asked to represent the pattern with concrete materials, describe the pattern in their own words, and reconstruct the pattern using another method. The aim of requiring the students to remake the pattern using a second method was to ascertain their ability to manipulate the visual pattern in a meaningful way.

A computer turns the number in the top row into the number in the bottom row.

Input	1	2	3	4	5	6	7	8
Output	4	7	10	13	16	19		

Figure 2 Table used in the semi-structured interview.

Students were asked to continue the pattern for two more steps, describe the rule the computer was using, and describe the ways that this task is similar to the patterning task.

All four groups of students were presented with both tasks. The following section presents relevant exerts from students' responses to the two tasks. It is believed that these exerts demonstrate how the differing reasoning processes aid reaching generalisations and linking representations.

The patterning groups and the patterning task

All students in the 100th percentile patterning group correctly identified the pattern. Elizabeth (100th percentile) quickly recognised the structure of the pattern and said *you start off with 4 sticks and make a square to get another square, you put one on top and one on the side and one on bottom. You don't have to put one there (the left) as you've already got one and then you put another one the same way as you did the first one.* Students in the 0th percentile patterning group experienced difficulty in initially seeing the pattern. For example, Alison (0th percentile) said *each time the first square is four and you keep adding four on and so I got up to 28.* Chris (0th percentile) said *Well with the three squares there was 10 sticks altogether so for you times ten for each three so........*

All the students in the 100th percentile patterning group were capable of reconstructing the pattern and describing the pattern in a variety of ways. For example, Bede in his second attempt said *You could take all the centres out (leaving the 1st and last verticals) so that you get a rectangle and then fill the sides in to make squares.* Not only could they recreate the pattern in a number of ways but also could express each algebraically and recognise that the expressions were equivalent When Bede expressed his pattern in symbols, he changed his strategy in order to make it easier to symbolise, "x x2 + (x+1)". None of the students in the 0th percentile group seemed capable of deconstructing and reconstructing the pattern. A typical response was *You can't do it in any other way.*

The table of data groups and the table of data task

Macarenya (100th table percentile) quickly said *you multiply the top number by 3 and add 1. I found this by looking at 6 and 19. The closest number to 19 is 18 so, 6x3+1.* Sarah (100th percentile) picked out pairs of data and tried a number of strategies in endeavouring to link them. All students at the 0th percentile suggested that you continued the patterns by going up in threes. Matthew (0th percentile) said *the top line goes up in 1, 2, 3, and so on but the bottom line goes up in threes so I don't think there is a rule because the top row doesn't have any relation in numbers between the top and bottom rows.*

Comparing the groups

The 100th percentile table and patterning groups shared many common characteristics. Both groups searched for generalisations early in the interview. Only one student in the 100th percentile table group could identify the correct generalisation in the patterning tasks and was capable of manipulating the concrete materials to reform the pattern in a new way but failed to identify that the two expressions were equivalent. By contrast all students in the 100th percentile patterning group were successful at the table task. To be successful at the patterning component it seems that students required some added characteristics above and beyond those exhibited by their table cohorts. They needed to be able to continually manipulate, both physically and/or visually, the materials to form new generalisations. This ability allowed them to see the common structures between the patterning question and the table question. They were the only group who could successfully map one problem onto the other. A typical response was *the input number in the table is the same as the amount of squares in the pattern and the output number is the same as the number of sticks,* thus successfully mapping the relational components of each question. The 100th percentile table group was unable to do this.

Discussion and conclusions

Spatial visualisation and patterning could all be seen as different dimensions of spatial ability. While spatial ability has been acknowledged in specific areas in the algebraic domain (Kirshner, 1989; Jurascheck & Angle, 1986), its role in beginning algebraic learning does not seem to have been addressed. The results of this study indicated that an ability to reason visually is significantly correlated with most early algebraic experiences, especially when generalising from visual patterns, and generalising from tables of data. When using the visual patterning approach, successful students manipulated the materials to form new patterns, broke the pattern into repetitive parts and reconstructed the visual pattern in a variety or ways, that is, mentally and physically manipulated, rotated, twisted and inverted the stimuli (Tartre, 1990).

The pattern reasoning process seemed essential for introducing the variable using visual patterns. The differing success between the 100th percentile patterning group and the 100th percentile table of data group with linking the two representations seemed to indicate that the pattern completion reasoning process could play a role in seeing the common structure between various representations. This conjecture needs further investigation. This difference could also be a result of the 100th percentile patterning students exhibiting a high degree of flexibility in their thinking. They exhibited an ability to perceive figures from different perspectives (Presmeg, 1986) and a willingness to change their approach to the solution (Krutetskii, 1976).

Research in the algebraic domain has tended to mirror and extend research carried out in the arithmetic domain, a search for structure and sequence (Boutlon-Lewis et al). The successful transition from arithmetic thinking to algebra thinking seems to involve more than generalising arithmetic. It also includes contextualising algebraic functions, and linking and transferring between differing representations (e.g., symbolic, graphical, verbal, visual). This research begins to indicate that the ability to visualise, particularly spatial visualisation and pattern completion, can aid students to distill the essence from the approaches commonly used. The ability to visualise may not only provide strength for developing spatial awareness but also may assist students in making sense of all the representations we commonly use when modeling mathematical ideas.

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LINGUISTIC ASPECTS OF COMPUTER ALGEBRA SYSTEMS IN HIGHER MATHEMATICS EDUCATION

Carl Winsløw, Department of Mathematics,
Royal Danish School of Educational Studies

The main contents of this paper is a theoretical analysis of actual and potential linguistic functions[1] *of Computer Algebra Systems*[2] *(abbreviated CAS) in the teaching of mathematics (mainly at tertiary level*[3]*). At the end, an ongoing experimental project based on this analysis is described briefly*[4].

1. Introduction.

The use of computer technology at all levels of mathematics teaching has been given much attention by recent PME studies[5]. The main focus has of course been the ways in which the use of computers affect the *thinking* of students and teachers, to the extent this may be observed through their interaction, through interviews etc. Also a socio-cultural point of view has been used to describe the interaction (in the precence of computers) from a more global perspective[6].

In mathematical activity in general, we obtain knowledge of the thinking of others from their (spoken or written) utterances; thus a main part of the activity is based on language use. To understand the ways in which the activity is changed in the presence of new technology, it is therefore necessary to analyse in what way the language use is affected by that presence, and of course this in turn presupposes a theoretical framework for describing language use (particularly mathematical). This paper is an attempt to provide such analysis in the case of CAS use at tertiary levels of mathematics education, based on the central notions of medium and register.

2. Roles of the computer: tool, medium and agent

Historically, computers were created as *tools* to carry out (large) computations, mainly for the purposes of technology, commerce and science (Hillel, 1993, Sec. 3.2). The discussion about to what extent the use of these tools may replace traditional skills continues to divide public, educators and politicians. To some degree, it is relevant also for more advanced mathematics teaching as dealt with here, but our main focus will be on two less debated aspects of the use of computers at this level: its roles as a *medium* and as an *agent*.

My analysis of the actual and potential role of CAS in mathematics learning is based on a linguistic view of mathematical knowledge and its learning (further exposed e.g. in Winsløw, 1999). According to this view, mathematical knowledge consists essentially in a certain type of linguistic competency, in particular, the ability to use symbolic and natural languages in a very specific regulated way – what linguists call a *register*[7] (regardless of the languages involved). It consequently views mathematics learning as based on the acquisition of this competency through participation in various forms of discourse where this register occurs. From this viewpoint, the computer initially presents itself as a *medium* for such discourse, i.e. on a par with other means by which communication in the mathematical register may take place, primarily speech and writing. It is a salient feature of many mathematics related programs to enforce certain changes in the formal rules governing the register (especially in the syntax of symbol strings); but in modern CAS, such changes are usually minimised, and it is a widely accepted assumption that the medium should not affect the register unnecessarily.

More importantly, in all interesting uses of computers for this purpose, it is more than a medium: it may also, in its own way, act as an *agent* in discourse. For instance, any spreadsheet or CAS – or even a pocket calculator – enables[8] the 'dialogue'

1+1=2

with the string '1+1=' being produced by the user, and the last symbol '2' being returned by the devise. This peculiar mixture of medium and agent is a chief object for our analysis.

3. The CAS as a medium.

Mathematical communication occurs chiefly in oral and written media, but unlike many forms of communication in natural language, oral communication is usually accompanied (if not dominated) by written components. This is especially so in the context of higher mathematics, with its extensive use of symbol strings; but even in elementary arithmetic or geometry settings, written symbols and figures[9] are very common parts of communication. The computer as we know it today presents itself mainly as a written medium, fully able to represent mathematical texts.

The mediator roles of a CAS fall in three main groups, depending on the agents involved in communication (ignoring, for the moment, the CAS itself):

- *Individual CAS work.* This is probably the most common use of CAS: one person works on one or more documents in CAS, without interacting with other persons.
- *Co-operative CAS work.* Two or more people work on the same document in CAS, either from the same computer (in front of which they sit together), or from networked computers. In the latter case, the contribution of the persons to the communication may or may not be real-time.
- *Ostensive CAS work.* One person writes or displays CAS documents which are visible to other persons (e.g. through networks or on a large screen) who are not able to communicate using the CAS (but may, or may not, write or speak about what they see).

These types of communication situations are different in many respects, as summarised in Fig. 1.

Especially the form in which the mathematical register occurs deserves some comment. Individual usage of CAS requires in principle mastery of formal aspects of the relevant parts of the register and the way it is implemented on CAS (these days, the latter is often largely self-explanatory), since it is only possible to use the system through formal writing (discounting here simple text editing as use of CAS).

	Individual CAS work	*Co-operative CAS work*	*Ostensive CAS work*
Typical purpose	Inquiry, or preparing exposition	Joint inquiry, Exchanging ideas	Demonstrating CAS or math. principles
Media	CAS	CAS, oral, email...	CAS, oral, email...
Direction of CAS communication	A to A or A to potential reader	A to others, with A changing	A to others, with A not changing
Math. Register	Formal, prerequisite	Formal and informal, usually prerequisite	Formal and informal, may be taught

Fig. 1.

Communication in the other two cases involves these formal aspects for the same reason, but because of the presence of other media, informal usage and meta-discussions may occur as well. In co-operative CAS work, formal competencies are usually assumed at a certain common level, in lack of which the communication may momentarily have to switch to modes similar to the ostensive. In any event, computer algebra systems are not designed to bridge the gap between formal and informal mathematical language use, so for this purpose, other media are needed.

4. The CAS as an agent.

What makes the different forms of CAS mediated communication radically different from its 'conventional' counterparts is the fact that the CAS 'acts' communicatively. It obviously does so in automated, non-intelligent ways – but it may still be perceived by the users as an active participant in the dialogue, and indeed the discourse produced by the system (output) will usually influence the subsequent communicative acts by the users (input or dialogue among users). In a recent British analysis of the role of microcomputers in undergraduate teaching, this phenomenon was characterised as *the independent personality*[10] *of the micro plus program in the student's eye*, and it was even qualified as *an invaluable illusion* (Burn et. al., 1998, p. 55). This illusion has interesting effects in all three cases considered before. For the individual 'scribbling' or 'computing' activity, the CAS can be experienced as a reliable[11] companion who takes care of the 'dirty work' and thereby frees the mind to focus on the overall purpose of the activity. In the ideal case, this may turn the 'communication with one-self' situation towards patterns that resemble the co-operative mode. In the context of co-operative work, the CAS may serve a similar 'assistant' function, but it may also act as an amplifier of the dialogue among speakers, e.g. through illustrations and manipulations thereof. However, it is also possible that the use of CAS turns the attention of the participating persons so much towards the management of input and output to the CAS that the whole situation approaches the individual scribbling mode[12] (where each participant is absorbed in private interaction with the system); or, if this management is left to one participant (which may be almost inevitable if the interaction takes place from a single computer), the communication may become ostensive rather than co-operative. Finally, the introduction of a 'third party' in the ostensive mode of communication may shift the perceived roles and authorities of the agents; for instance, a lecturer and his students may feel (and act) as 'fellows' when prompting and interpreting output from the CAS, thus turning the ostensive mode at least partially towards the co-operative[13]. We may illustrate the possible effects of the CAS agent on the initial (and perhaps intended) communication mode as follows:

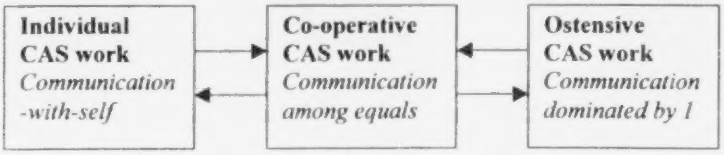
Individual CAS work
Communication -with-self
Co-operative CAS work
Communication among equals
Ostensive CAS work
Communication dominated by 1

Fig.2

The pedagogical problem with these possible shifts is that, whether or not they are desired, they may be difficult to control. The apparently innocent (impersonal and will-free) CAS agent may be experienced to exercise a considerable influence on the way communication is organised.

The participation of the CAS agent in mathematical communication has other serious potential effects as well (at least with present technology). The most obvious is the framing of subject matter that occurs in a CAS-mediated communication: even within the formal setting, the mathematical capabilities of the agent are by necessity limited. Natural 'higher' levels of abstraction may typically not be articulated.

5. The CAS as a tool for teaching and learning.

After examining some general features of CAS mediated mathematical communication, let us now consider its functions in teaching higher mathematics from a general perspective. First of all, the desired functions will depend on the motives behind using a CAS (Hillel, 1993), and at least in the setting of university teaching, there are two main types of arguments:

1. The widespread use of CAS in advanced mathematical practice at large (academic, technological and so on) justifies and necessitates their appearance in advanced mathematics teaching;
2. CAS can be used to create new and powerful ways of learning advanced mathematics.

Both arguments are quite general and, as they stand, non-specific; they thus represent two main classes of motives for using a tool[14] in teaching a subject for the practice of which this tool may be used. The point is that they have, in principle, different effects. If the first one is dominant, then the tool becomes itself part of the subject matter, which is therefore substantially changed. If the second one is dominant, the subject matter is a priori the same, although changes in emphasis may appear natural or even inevitable in presence of the tool; a relatively innocent example is that if the tool facilitates the learning of some parts substantially, others may be given more attention. In the first case, the function of the tool is to be itself an object of learning; in the second case, it is to support learning. We mention this distinction here to point out that the goal of our investigation is mainly concerned with the second kind of function of CAS use. In the actual teaching project to be discussed later, this in fact has a double meaning: to facilitate the learning of the students, but also to provide the student with experiences that will be of value for use of CAS in their own teaching.

Settling with point 2. above as our main motive, what roles do we expect CAS to play in the students' learning and how do we implement them? I believe the first thing to realise is that our own picture of the learning process is heavily influenced by our own experience as learners, by our 'learning history'; and this typically does not involve the CAS as tool and medium. Of course, for the whole question to be meaningful, we do have experiences with CAS use in relation to the subject matter (typically as an instance of point 1.), but usually not with its use in learning the subject. Our learning history makes us readily follow Hilbert's description of mathematical signs as *marks on paper*[15] (some of us might add: and on blackboards). But how about dots on a screen? Or even built-in routines? The problem is partially mitigated by the fact that the CAS has been constructed by people whose learning history is similar to ours, and indeed they are designed to minimise our initial feeling of alienation. But this also accentuates the problem: the CAS has typically been designed to assist people with well-developed competencies in the mathematical register, not to assist people develop it. For instance, current systems provide little help if the input contains syntactic mistakes. Even more disturbing is the fact that output often requires substantial interpretation (such as symbolic transformations, possibly performed through new input to the system) or that it is a priori meaningless (for instance, if the CAS cannot carry out a required operation, it may just reformulate or repeat the input). This suggests that the role of CAS in student's learning must be, at least initially, a complementary medium (with built-in agent).

As a *medium*, there are several ways in which CAS could be expected to provide learning opportunities not otherwise available in written communication. Most eye-catching is the ease and elegance with which algebraic and geometric inscriptions may be processed and combined; the functions of this material (and partially 'aesthetic') aspect are perhaps not sufficiently investigated. In daily student work, another important feature is that patterns of problem solving may be repeated by changing only a few parts of the input, thus highlighting the pattern and avoiding repetitive writing. But what appears to me to be the most striking advantage is that the fundamental distinction of symbol language and natural language (Winsløw, 1999) is an explicit and organising principle for communication in this medium (unlike in speech or ordinary writing, where the distinction has to be established analytically). Namely, 'text' and 'symbol strings/geometric figures' have entirely different status, being handled by separate devices that may be viewed as two channels of communication within the medium.

It should be noted here that the way this linguistic principle is built in leaves much to be desired, because the *interaction* of natural and symbolic language (which is just as fundamental to the mathematical register) is not supported by current CAS understood *as agents*. In this respect, the CAS offers nothing more than conventional writing. That is, CAS does not in any way react to natural language text, even in strictly logical use – and then of course it also does not offer ways to handle even simple combinations with symbol language text (such as 'It is false that 1=1').

As an *agent*, CAS enables certain types of example investigations which would otherwise be painstaking or impossible, but which could reveal new aspects of the subject matter. Especially higher complexity (not to be confused with higher abstraction) in discourse is greatly facilitated when using CAS as an 'assistant'; this may be particularly desirable if realistic mathematical modelling is part of the teaching agenda (Blomhøj, 1998). The ease with which examples can be generated and examined may help to tackle the classical problems with abstraction where students are in lack of sufficient cognitive 'roots' (Skemp, 1987, Sec.2), especially for concepts where substantial examples are hard to investigate by hand (e.g. unstability phenomena in dynamical systems). Another potential is greater flexibility in the structuring of teaching; for instance, a method may be initially presented as a 'black box' computer routine, with only input and output understood by the students, prior to a more theoretical treatment (Hillel, 1993, p. 36). Finally, in versions of CAS which allow programming new routines, each of these functions of the agent may be improved not only in scope but also in quality, because programming routines (rather than applying built-in ones) may be a way to develop a structured understanding of concepts[16].

The second question, of implementation, is informed but not answered by these considerations. As pointed out by Hillel (1993), general claims and beliefs about the functions, benefits or pitfalls of computer usage in mathematics teaching can *only be evaluated when...embedded in an educational setting...which includes textbooks, tasks and the social environment*. I present, at the end, such a setting.

6. A teaching project using Mathcad for 'Calculus with analytic geometry'.

I run a course entitled 'Analysis and Geometry' for students preparing their Masters Degree in mathematics education. The students are all practising teachers at various secondary levels. The aim of the course is twofold: to teach the subject matter (multivariable calculus and differential/analytic geometry), and to promote reflection on pedagogical and philosophical aspects of (elementary) analysis teaching.

My intention is that bringing in CAS (Mathcad) use as an integrated part of the course should serve both aims. The potential for supporting subject matter learning was discussed extensively above. But also our own lack of 'learning experience' with CAS was mentioned, and it is a main point of investigating this particular setting that the students are both experienced teachers and active students (in the course). Their reflections on the use of CAS is thus in a unique way focused on *both* teaching and learning. To maximize the benefits of this situation, the students are videotaped while using CAS in the three modes listed above, and later some will be offered help to integrate CAS use in their own actual teaching (at mathematically more elementary levels). The students continuously deliver logbooks (in Mathcad) and are interviewed regularly.

Although the theoretical analysis of this paper has not been unaffected by the first experiences from the above project, one difficulty has to be mentioned: It seems quite difficult to sustain *cooperative* CAS work. Due to differences in the cooperator's background knowledge, such work tend to switch to individual mode (one agent interacts with the CAS while the others loose interest) or to the ostensive mode (as before, but retaining the attention of the group). It also seems that it is knowledge of the medium (computers in general, Mathcad in particular) rather than knowledge of the register (mathematical competency) which determines who dominates the communication. Indeed, there is a substantial difference in individual student performance from activities (say exercise solving) based on the CAS medium to activities using conventional media. It is most interesting to see whether this apparent difference in 'performance quality' is upheld or disappears as the students become more familar with both Mathcad (in learning and, later, teaching) and the subject matter.

Notes.

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[1] Roughly in the sense of Jacobsen (1967).

[2] We shall not attempt to give an abstract definition of a CAS. By a CAS I mainly think of the programs Maple, Mathematica, and Mathcad, however, the analysis undoubtedly remains valid for usage of other 'similar' software.

[3] As regards empirical grounding, I am presently mostly working with the introductory tertiary level, particularly the context of calculus. The main reason for this is that at present CAS cannot handle the central items of more advanced topics (such as abstract algebra or functional analysis).

[4] The talk will probably expand this part further.

[5] By rough counting of papers presented, the topic 'Computers, Calculators and other technological tools' was the largest at PME 23 (see proceedings, p.xxxviii).

[6] See e.g. the articles of Chronaki and Gardiner in the proceedings of PME-23.

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[7] For more on this notion and its use, see (Halliday et. al., 1964), and (Pimm, 1986).

[8] Of course, modulo the mentioned deviations in the syntax of symbol strings in input and output which may be enforced by the system.

[9] Instead of written icons and symbols, one may also use concrete physical objects (e.g. models of geometric figures or simply 'real life objects'), and then mathematical communication may include various forms of gesturing (pointing, streching etc.) relative to these. This is, however, not different in principle from what occurs in relation to the physical manifestations of writing (e.g. chalk on a blackboard).

[10] It is an interesting idea to associate 'personality' with CAS, because a priori the lack thereof seems to be one of its characteristics as an 'agent'.

[11] Reliable – with obvious but serious restrictions: no room for imprecision in input, and more than potential risk of absurd output if the system's 'knowledge area' is transgressed.

[12] With the modulations resulting from CAS (mentioned above) slightly counteracting this move as well.

[13] Conversely, the intervention of an instructor in student's co-operative work may in my own experience be felt less as the intrusion of an authority in the presence of the 'CAS agent'. In other words, it may ease turning the co-operative mode towards the ostensive when so desired.

[14] At this point, we may consider a medium as a 'tool for communication'.

[15] Quoted in N. Rose: *Mathematical Maxims and Minims*, Rome Press Inc., 1988.

[16] Of course, there is a risk that the technicalities of programming take a good deal of the attention; but the development of programming languages closer to the 'mathematical register' seems to be well on its way.

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TWO PATTERNS OF PROGRESS OF PROBLEM-SOLVING PROCESS : FROM A REPRESENTATIONAL PERSPECTIVE

Atsushi YAMADA
Department of Mathematics Education,
Aichi University of Education, JAPAN

ABSTRACT

In this article, two patterns of progress of problem solving are indicated from a descriptive framework that solver's internal problem representation is set beside with solver's external activity, which is also based on both Goldin's "a model for competency in mathematical problem solving" and a representational perspective that problem solving is the process that solver constructs and transforms her/his internal/external representations. Also, to explain one side of the patterns, a problem solving process is analyzed. And based on the analysis, a feature of the problem-solving process and the difference between the two patterns are discussed.

1. BACKGROUND

A series of my studies aim to build a theoretical model to describe cognitive processes in mathematical problem solving, which are based on a perspective that problem solving is the process that solver constructs and transforms her/his internal/external representations. In this article, we will deal with patterns of progress of problem solving from this perspective.

In many cases, representations that solver mentally constructs from problem statements have various aspects, and they change with progress of problem solving. For example, representations constructed just after reading problem statements should be mainly verbal/linguistic representations, but at its final stage of problem solving they may differ from the previous one in many ways (for example, formal notational representations may be added to them). In my studies, internal representations that solver mentally constructs from problem statements and transforms with progress of problem solving are dealt with synthetically, and the integrative internal representation that is called "problem representation" (Yamada,1997). And, in order to interpret various aspects of problem representation, my studies are based on G.A.Goldin's "a model for competency in mathematical problem solving" (e.g., Goldin,1987;1988; 1992). Because Goldin's model has five cognitive representation systems (verbal/syntactic; imagistic; formal notational; planning, monitoring, and executive control; affective) and each one may be considered as a generator of components of problem representation, that is, it gives us five viewpoints to discuss the solver's internal problem representation. Furthermore, another descriptive viewpoint, the external solver's activities(including external representation), should be set beside the internal problem representation to interpret problem representation more closely. Here we have two points of view to interpret problem-solving process; i.e., internal problem representation and external solver's activity. And we can consider the progress of problem solving as the interaction between internal problem representation and external solver's activity.

Although in mathematics education many studies on mental representation have been made over the last few decades, only few attempts have been focused on the interaction problem representation and external solver's activity during problem solving so far. But several articles have been recently devoted to the study of the way that solvers construct/transform their problem representation in problem solving situation (e.g., Cifarelli,1998). Based on the above perspective, in this article, we will deal with a research question about the patterns of interaction between "problem representation (and its transformation)" and "solver's external activities", which will indicate how problem solving progresses. To put it more concretely, this article aims to indicate two concrete patterns of progress of problem solving from the above perspective.

2. TWO PATTERNS OF PROGRESS OF PROBLEM SOLVING

2.1. *Theoretical Derivation of Two Patterns of Progress of Problem Solving*

Simply considering, we can infer that problem solving makes an essential progress when solver's activity greatly changes. In this sense, we may use well-known theory about phases in problem solving. For example, Polya's four phases, "understanding, planning, execution, looking back" (Polya,1956) or the labels of Schoenfeld's episode analysis (Schoenfeld,1985) are characterized by the difference between the problem-solving activities, and they could be considered as the labels for description of progress of problem solving. But, from my study's descriptive framework that set problem representation along with solver's activity, we can suppose two patterns of progress of problem solving, i.e., two patterns of interaction between problem representation and solver's problem-solving activity. Fig.1 schematically shows them. One is "change of solver's activity (A1 -> A2) caused by transformation of problem representation (R1 -> R2)" shown by solid arrows (I will use "Pattern [I]" to refer to this pattern in this article), and another is "transformation of problem representation (R1 -> R2) caused by change of solver's activity (A1' -> A2)" shown by broken arrows (I will use "Pattern [II]" to refer to this pattern in this article).

Those patterns are theoretically drawn from my study's descriptive framework. But, how do they appear in concrete problem-solving process ? Therefore, we discuss the two patterns along with concrete problem-solving process.

2.2. *Transformation of Problem Representation Changes Solver's Activity*

When we pay attention to solver's cognitive aspects, we may naturally consider transformation of solver's problem representation as a cause of change of solver's

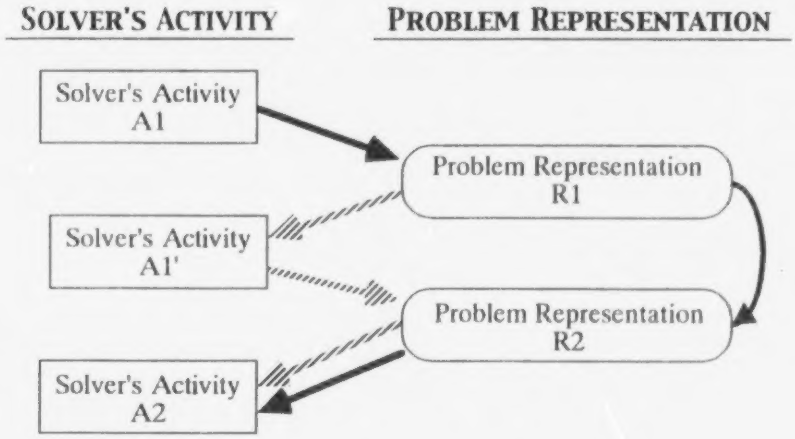

Fig.1: Patterns of Interaction between Solver's Activity and Problem Representation

activity. Here, let us consider typical examples of this pattern of problem-solving progress.

First, there is a case where solvers transform their *part(s) of* problem representation. We suppose that verbal/syntactic, imagistic, and formal notational systems of cognitive representation in Goldin's model correspond to some of the external representational systems such as Lesh *et al.*(1983) or Ishida(1984). Therefore, if a part of problem representation which correspond to an external representation is mentally transformed, the external one will naturally change. The change of external representation will be often attended with change of solver's activity. For example, just after reading a word problem that bring simultaneous equations (A1), a problem representation that solver constructs seems to be occupied by verbal representations (R1). But, if the solver can grasp semantic/relational structure of the problem, the problem representation will include imagistic or formal notational representations (R1 -> R2). At this time, the word problem will be translated into simultaneous equations, and then the equations will be solved (A1 -> A2). To borrow the labels of Schoenfeld's episode analysis, this change of solver's activities is described as "from Understanding to Planning-Execution". Anyway, it seems likely that transformation of the parts of problem representation corresponding to external representation such as language, image/diagram, formal notation *etc.* causes the change of solver's activities.

Secondly, we shall focus on solver's goal. One may certainly say that change of solver's goal causes solver's activity, because solver's goal directly guides her/his problem solving to be consistent in a phase of problem-solving process. We have a theoretical ground for thinking so and for relating solver's goal with problem representation, but to think so, we have to explicate Goldin's model in further detail.

Goldin's model includes a cognitive representational system of "planning, monitoring, and executive control" that direct or guide the problem solving process (Golin,1992; 1998). And, Goldin(1998) supposed that "this system includes competencies for: (1) keeping track of the state of affairs in the other systems, and in itself; (2) deciding the steps to be taken, or moves to be made, within all of the internal representational systems, including itself; and (3) *modifying* the other systems"(p.153). In a phase of problem-solving process, this system may serve as a system for generating and transforming the solver's goal because it especially includes the competence (2) that is strongly related to solver's goal to guide her/his problem-solving process. And, this representational system may be also reflected in solver's mental problem representation at that time. Therefore, it seems likely that transformation of a part of problem representation related to this representational system, i.e., transformation of solver's goal, cause the change of solver's activities. Again, take the above word problem for example. If the solver feel difficult to solve the problem with simultaneous equations, and if her/his goal, "to solve the word problem with simultaneous equations", change to such as "to try to draw pictures" in her/his problem representation (R1 -> R2), the previous solver's problem-solving activity changes "Planning-Execution" into "Analysis" with drawing picture (A1 -> A2). Added to this simple example, we may consider problem solving as a series of processes of change of solver's goals, and can actually describe it just so (Yamda, 1998). This result seems to be a good illustration of this pattern of progress of problem solving.

After all, both of above two patterns transformation result in the pattern of "transformation of problem representation changes solver's activity". Next, we will take the reverse pattern.

2.3. *Solver's Activity Changes Problem Representation*

Second pattern shown broken arrows in Fig.1 is the following: a certain solver's activity (A1') causes a transformation of problem representation (R1 -> R2). By the way, in most cases, the activity (A1') is based on certain problem representation (R1), and the transformation of problem representation (R1 -> R2) is observed by change of solver's activity (A1' -> A2). Therefore, all of them (R1 -> R2 and A1' -> A2) are shown in Fig.1. To think of this pattern concretely, we shall take the following word problem given to a 6th grader, S, and his problem-solving process for example.

Problem

At a restaurant, if eating 100 dumplings within 5 minutes , the price becomes free. Masakazu tried this special challenge, and he was able to eat 100 dumplings just 5 minutes. He had a strategy to accomplish it ! The strategy was to eat 6 fewer dumplings in a minute than in the previous minute. How many dumplings did he eat in the first minute?

Outline of solution process (Solver S's solution)

At the beginning of solution process, S could not understand a part of the problem sentence "... to eat 6 fewer dumplings in a minute than in the previous minute", and didn't have any concrete image of the problem situation. However, when interviewer brought up 100 coins, he began to understand gradually the problem situation using those coins. After a while, he chose 30 coins for initial term for a sequence of number of dumplings, and tried an attempt to pick up coins like 30, 24, 18, 12, 6. Then the activity had stopped soon, but he only noticed that the rests of coins were 10.

After that, S was encouraged to draw figures by interviewer, but again he began to repeat picking-up-coins operation muttering, "How many should he eat...?". When interviewer intervened saying "How about 40?", he immediately reacted to the interviewer saying "No !". Thus, he intuitively noticed that the temporary initial term of sequence of dumplings was between 30 and 40, and actually confirmed it with picking up coins. But his activity changed dramatically at this time. Instead of the picking-up-coins operation, his activity became a systematic construction of sequence(arithmetical progression) using guess and check with conviction. He could smoothly narrow the temporary value for initial term with 36, 30, and got the right answer, 32.

Guess and Check is one of the most effective strategy to solve this problem. In the latter half of this problem-solving process, it seems that the solver S adopted the strategy[*] twice: first in the phase of picking-up-coins operation, secondly in the phase of construction of sequences. Although the strategy used in those phases was consistent, the piking-up-coins activty (A1') drastically changed into the constructing sequence activity (A2) in the last phase. During the picking-up-coins activity, it seems that his problem representation was dominated by kinesthetic/operational imagistic representation (R1) and guided his activity (A1'). After keeping on the picking-up-coins activity, however, he suddenly began to use an abstract sequence of number without using coins (A2). At that instant, he could abstract a pattern of arithmetical progression from repeating picking-up-coins operation and transform the previous problem representation into the more abstract one (R1 -> R2) that led him to new activity (A2).

Thinking of this case, we may consider that the problem solving progressed owing to *repeating* picking-up-coins activity rather than autonomous transformation of internal representations, because before constructing sequence of number the solver did not have any reliable problem representation that would have guided him to successful solution. That is to say, the solver's activity caused the transformation of problem representation and progress of his problem solving.

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[*] Since he neither fully understand the problem structure nor effectiveness of the strategy until the phase of construction of sequence of numbers (arithmetical progression), maybe I had better call the way he adopted there a naive methodological idea, not a strategy. How to call it, however, is not important here, thus I will temporarily use the word, stratetgy.

3. DISCUSSION

3.1. *What is the Difference Between the Two Patterns?*

At first glance, the above two patterns of progress of problem solving seem to be alike because both patterns finally result in a process that a problem representation (R2 shown in Fig.1) leads the previous solver's activity to a new activity. But it is an argument only from observer's side, and takes no account of the cause of the activity's change. From my study's representational perspective, Pattern [I] and [II] are unlike. In Pattern [I] an autonomous transformation of solver's problem representation (R1 -> R2) causes a change of solver's activity (A1 -> A2) and the previous activity A1 is not so involved in the transformation of problem representation in most case, whereas in Pattern [II] a solver's activity A1' causes a transformation of problem representation (R1 -> R2) and the emergence of new problem representation R2 is greatly involved in the solver's activity A1'. Certainly, what we can observe is mainly solver's external activities, but it is reasonable to search the solver's internal/cognitive factors for causes of progress of problem solving when interpreting a problem-solving process. It is very different viewpoint whether progress of problem solving is caused by the solver's external factor or internal/cognitive factor.

Here, we should notice the Pattern [II]. From a narrow representational perspective that think little of solver's external activity and interaction between it and internal representation, the mental construction and transformation of problem representation (also including the integration of parts of problem representation) is tended to be consider as the prime cause of progress of problem solving. But, as has been suggested above, there seems to be many cases that an external solver's activity, even if it is not goal-oriented, lead to creation/abstraction of new problem representation. This is an important point to notice, because not all the internal representations (and also strategies) may result from mental activity, rather many of them may result from external activities. If a solver can construct an appropriate problem representation from a word problem or apply general complete strategies to her/his problem representation, it will typically lead to Pattern [I] and there is little scope for teaching and learning of problem solving. Of course that is a desirable status. But, in the context of teaching and learning of problem solving, it is important for teachers to discover and encourage students to use naive methodological ideas, which may become a powerful strategy through interaction with problem representation in future, such as the strategy solver S used in earlier phase.

Next, paying our attention to Pattern [II] we shall discuss more concrete pattern of progress of problem-solving process.

3.2. *A Feature of Progress of S's Problem-Solving Process*

As I said earlier, problem representation that solver constructs (and transforms) has various aspects, because we can suppose that problem representation has many kinds of component parts. Although five kinds of representation that directly related to representational systems in Goldin's model is supposed to be the components in this study, this issue is a little more complicated, for example, we can also suppose many kinds of imagistic representations in the imagistic representational system. Actually, Goldin (1998) supposes three kinds of imagistic representational systems: visual/spatial, auditory/rhythmic, and tactile/kinesthetic systems of representation. Therefore we may consider three kinds of imagistic representations as components of problem representation. Furthermore, Presmeg (1986) found five kinds of visual imagery used by high school students, which we may also regard as imagistic components of problem representation: concrete/pictorial imagery, pattern imagery, memory images of formulae, kinaesthetic imagery, and dynamic imagery. But, some of them overlap with each other, and we can also consider some levels of abstraction among the imagistic representations. For example, an internal representation that generates an arithmetical progression seems to be more abstract than a kinesthetic/operational representation that guide concrete picking-up-coins operation. This point of view might be anticipated in the above analysis of S's problem-solving process, but let us examine it again here from this viewpoint.

What made S's problem solving progress in last phase was to repeat picking-up-coins operation. As I suggested earlier, during the activity, it seems that his problem representation was dominated by kinesthetic/operational imagistic representation. And then, he could abstract a pattern of arithmetical progression from repeating picking-up-coins operation and transformed the previous problem representation into the more abstract one. It seems that this new problem representation was dominated by a kind of pattern imagery (Presmeg,1986) to be able to generate arithmetical progression (and still strongly related to imagistic representation). That is, abstraction of problem representation, putting it more precisely, abstraction of an imagistic representation of problem representation, occurred in this process. This is a significant feature of his problem-solving process and shows a concrete functional pattern (especially related to Pattern [II]) of interaction between solver's activity and problem representation.

4. SUMMARY

In this article, I suggested two patterns of progress of problem solving from a theoretical representational perspective, and described a problem-solving process with this perspective (and the conception of problem solving).

We may consider this attempt as an application of Goldin's model to description/analysis of problem-solving process, which is originally a model for description of competence in mathematical problem solving. To examine more problem-solving processes through Goldin' model or through the framework for description/analysis of problem-solving process in this article may help us to understand some concrete patterns of interaction between representational systems (and/or solver's activities).

Furthermore, a point of argument that abstraction of problem representation and/or solution activity may contribute to problem solving seems to be related to a finding of Cifarelli(1998), although his experimental setting that college students solved a set of similar word problems is different from this article's example.

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STUDENT OPTIMISM, PESSIMISM, MOTIVATION AND ACHIEVEMENT IN MATHEMATICS: A LONGITUDINAL STUDY

SHIRLEY M. YATES
FLINDERS UNIVERSITY ADELAIDE AUSTRALIA

Relationships between students' optimistic or pessimistic causal explanations, motivation, emotional well-being and achievement in mathematics were examined in a longitudinal study of primary and lower secondary students in South Australia. Mathematics teachers ratings of students' classroom behaviour and achievement were also investigated. PLSPath analysis indicated a predictive relationship between students' optimism and pessimism and motivation to their mathematics achievement. Students' prior motivation and achievement influenced teachers' judgments. Teacher ratings predicted students' subsequent achievement and related weakly to students' depression one year later, which linked in turn to their motivation and achievement. Implications for the psychology of mathematics education are discussed.

Introduction

Achievement in mathematics is influenced by many factors including the teacher, classroom environment, subject matter and students' motivation, psychological characteristics and prior knowledge. Students bring to school beliefs about themselves as learners, about learning and about the importance of school for them and their future (Paris & Newmann, 1990). Mathematics teachers encounter a range of students who differ in their reasons for learning and achieving academically, with many students holding strong and often negative views (McLeod, 1992; Yates,1997). As mathematics is a core component of universal compulsory education, it essential to gain an understanding of the beliefs that students bring to the classroom and their effects on students' views about mathematics, classroom behaviour and achievement.

Recent research has focussed on how people think about and explain the causes of everyday events in their lives, as these characteristic explanatory styles are related to health and achievement in work, sport and education (Peterson & Bossio, 1991; Nolen-Hoeksema, Seligman & Girgus, 1992; Yates, 1999). By the age of eight years, children have learned to make consistent attributions about the causes of events from predominantly optimistic or pessimistic mental frames of reference (Nolen-Hoeksema & Girgus, 1995; Yates, 1998a). Optimists view the causes of positive events as long term, generalisable and due to their own efforts and negative events as being temporary, specific and not their fault. The reverse is true of pessimists who interpret negative events as permanent, personal and pervasive and positive events as transient, ephemeral and external. Pessimistically oriented students are more likely be depressed (Seligman, 1990), to discount their successes, and when confronted with failure in the classroom, to give up more easily. In response to repeated failures, they display characteristically passive learned helplessness behaviours and reduce effort, stop trying or simply opt out altogether. For pessimists, "nothing fails like failure".

Motivation has been described by Brophy (1998, p. 3) as "a theoretical construct used to explain the initiation, direction, intensity and persistence of behavior, especially goal directed behavior... It refers to students' subjective experiences, especially their willingness to engage in lessons and learning activities and their reasons for doing so". Research on achievement motivation has indicated that the goals that students espouse for their learning effect the quality of their motivation (Dweck, 1986), which in turn influences behavioural, cognitive and affective outcomes (Urdan, 1997). Students' task involvement and ego orientation goals are important indices of motivation, as they reflect students' reasons to achieve mastery of the subject matter, to be competitive or to do both (Nicholls, Cobb, Wood, Yackel & Patashnick, 1990).

Differences between task involved and ego oriented students have been found in the time spent on learning tasks, persistence in the face of difficulty, quality of engagement, and use of adaptive mental strategies (see, review by Ames, 1992). Students who endorse task involvement learning goals demonstrate a range of adaptive behavioural responses (Pintrich & De Groot, 1990), while ego oriented students are more likely to exhibit maladaptive learning behaviours including the adoption of some helpless responses (Ames, 1992). Learned helplessness, as rated by teachers on the *Student Behavior Checklist* (Fincham, Hokoda & Sanders, 1989) is related significantly to students' general achievement (Nolen-Hoeksema *et al.*, 1992).

Students have exhibited learned helpless behaviours in mathematics classrooms (Yates, 1998b), particularly in situations of actual or conceivable failure. However, it is unclear whether the manner in which students ascribe the causes of everyday events influence their goal orientations and their achievement in mathematics. Observations from teachers are also important as they are likely to be cognisant of at least some of the recognised aspects of helplessness as they surface in classroom life.

Aims of the Study:

The aims of this study were to examine relationships over time between:

1. students' optimism, pessimism, motivation and achievement in mathematics;
2. teachers' perceptions of students' achievement and behaviour in mathematics classrooms and students' optimism, pessimism, depression, motivation towards and achievement in mathematics; and
3. students' gender, grade level and school and their motivation and achievement.

Research Design

Longitudinal data were collected on 243 students initially drawn from two primary schools, but spread over 26 primary and 24 secondary schools three years later. Students were administered a standardised test of mathematics achievement and questionnaires about their optimism, pessimism and motivation towards mathematics in the first and third years. Self-reported depression was measured also in the third year. Teachers rated the students' achievement and classroom behaviour in the second year of the study with a checklist designed to measure learned helplessness and mastery orientation in mathematics. The factors measured and the timing of these factors within the study are presented in Figure 1.

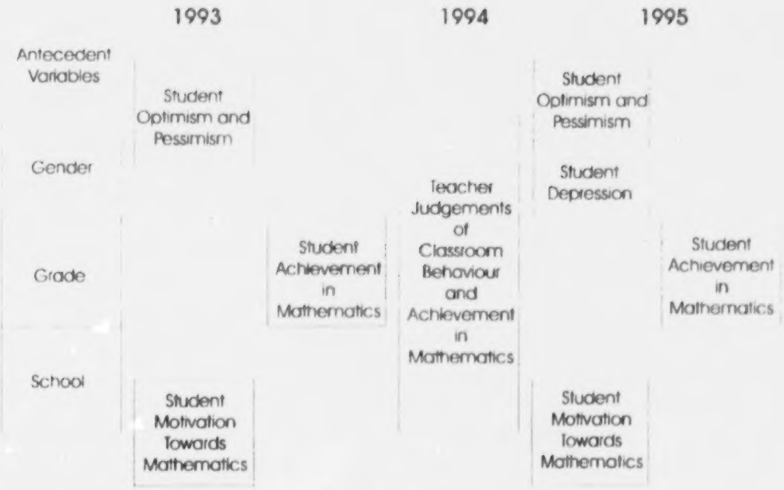
1993
1994
1995
Antecedent Variables
Gender
Grade
School
Student Optimism and Pessimism
Student Motivation Towards Mathematics
Student Achievement in Mathematics
Teacher Judgements of Classroom Behaviour and Achievement in Mathematics
Student Optimism and Pessimism
Student Depression
Student Motivation Towards Mathematics
Student Achievement in Mathematics

Figure 1 An overview of the factors in the longitudinal study

Measurement of the Factors

Student Optimistic and Pessimistic Explanatory Style: *Children's Attributional Style Questionnaire* (*CASQ*), (Seligman, Peterson, Kaslow, Tanenbaum, Alloy, & Abramson, 1984). Students chose between two possible explanations for each of 48 hypothetical statements about good and bad events.

Student Motivation towards Mathematics: *Your Feelings in Mathematics: A Questionnaire* (Yates, Yates & Lippett, 1993; 1995). This five point scale measured task involvement and ego orientation dimensions of students' goal orientation beliefs in mathematics. Each item commenced with the stem, *Do you really feel pleased in maths when ...* followed by a statement. Students' interest and engagement in learning mathematics was measured by task involvement items such as (Item 15), *something you learn makes you want to find out more*. Students' competitiveness was sampled by ego orientation items including (Item 23), *you score better on a test than others.*

Student Achievement In Mathematics: *Progressive Achievement Tests in Mathematics* (*PATMaths*) (ACER, 1984) consisted of three timed standardised multiple choice format (Form A) tests. In 1993 students were administered Test 1, 2 or 3 as recommended by the *Teachers Handbook* (ACER, 1984). In 1995 students in Grades 5, 6 and 7 took Test 1 or 2, Grade 8 Test 2 or 3 and Grade 9 Test 3.

Student Depression: *Children's Depression Inventory* (*CDI*) (Kovacs, 1992). Students rated one of three sentences for each of 26 symptom orientated items.

Teacher Judgments of Classroom Behaviour and Achievement in Mathematics
Mathematics teachers rated students' classroom behaviour on a five point 24 item *Student Behavior Checklist* (Fincham *et al.*, 1989). The items, selected from the research literature to measure learned helplessness and mastery orientation, included statements such as, *Prefers to do easy problems rather than hard,* measuring learned helplessness and, *Tries to finish assignments even when they are difficult,* for mastery orientation. Teachers gave a single estimate of student's mathematics achievement.

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Methodology

The *PATMaths*, *CASQ* and *Your Feelings in Mathematics: A Questionnaire* were administered in Term 1, 1993 to 335 students in Grades 3 to 7 in two metropolitan primary schools in South Australia. In November, 1994, 58 teachers in 31 schools completed the *Student Behavior Checklist* for 258 of these students who were then in Grades 4 to 8. In Term 4, 1995, the same instruments used in 1993 were administered, together with the *CDI*. The final sample of 243 students, presented in Table 1, were located in Grades 5 to 9 in 26 primary and 24 lower secondary schools.

Table 1 Numbers of students by Grade level and gender in 1993/1995

Gender	Grade 3/5	Grade 4/6	Grade 5/7	Grade 6/8	Grade 7/9	Total N
Male	8	28	21	28	24	109
Female	10	34	22	38	30	134
Combined	18	62	43	66	54	243

Methods of Data Analysis

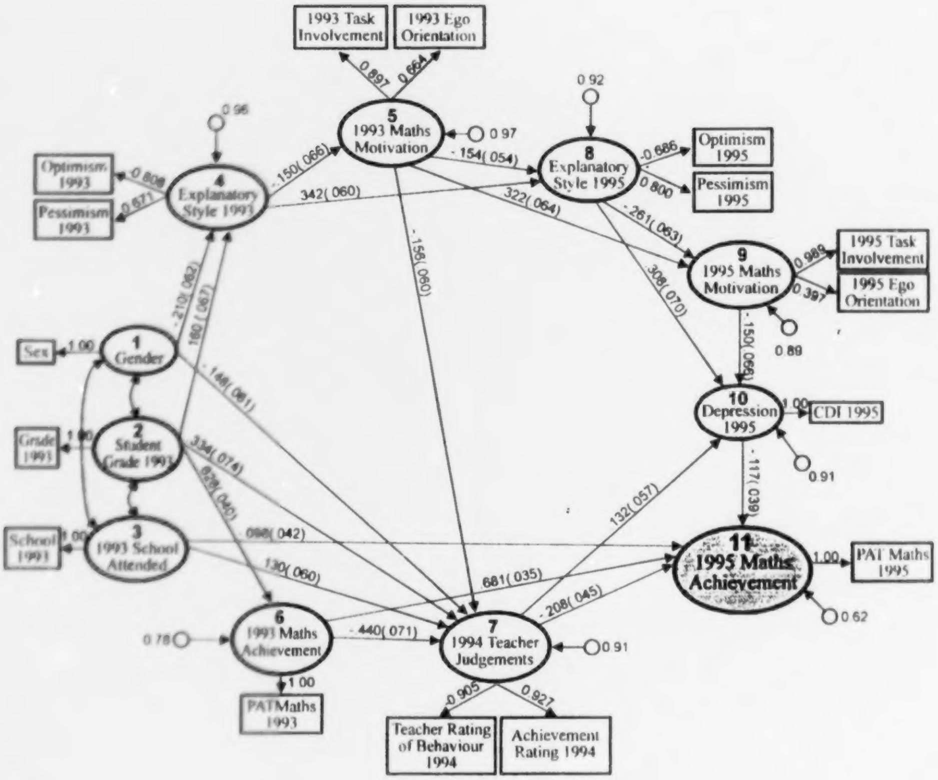
Rasch Calibration

All instruments were Rasch calibrated, to bring them to scales with common interval properties. While the *CASQ* and *CDI* met the unidimensionality requirements of item response theory (Rasch, 1966), confirmatory factor analysis of the *Student Behaviour Checklist* indicated a single ten item scale of *Academic Behaviour* (Yates & Afrassa, 1995) which was Rasch analysed. Factor analysis of *Your Feelings in Mathematics: A Questionnaire* indicated two scales of *Task Involvement* and *Ego Orientation* (Yates, 1997; Yates & Yates, 1996). These two scales were analysed separately.

Responses from the 243 students to the *CASQ*, *Task Involvement* and *Ego Orientation* scales were equated concurrently with pooled data (Morrison & Fitzpatrick, 1992). Students' raw *PATMaths* scores were placed on a single Rasch scale (*Teachers Handbook*, ACER, 1984), irrespective of the level of the test and the time of administration. All statistical analyses were based on Rasch case estimate scores.

Path Analysis

As the sample was non-random and non-representative, path analysis with latent variables, for causal modelling, was chosen as it did not require strict distributional assumptions and controlled for variables that might confound covariation patterns observed between variables (Tuijnman & Keeves, 1997). The observed manifest variables (MV) constituting the outer model were grouped meaningfully to form unobserved or latent variables (LV). Hypothesised relationships between the manifest and latent variables and between the latent variables of students' explanatory style, depression, motivation and achievement in mathematics and teacher judgments were tested. Significant relationships were then estimated with the jackknife technique (Tukey, 1977), using the PLSPATH program (Sellin, 1990). The influence of the antecedent exogenous variables of student sex, Grade and primary school in 1993 were also included. Figure 2 presents the significant paths and their standard errors.


1993 Task Involvement
1993 Ego Orientation
0.897
0.664
0.96
5 1993 Maths Motivation
0.97
0.92
Optimism 1993
Pessimism 1993
0.808
0.671
4 Explanatory Style 1993
-.150(.066)
.342(.060)
-.154(.054)
8 Explanatory Style 1995
-0.686
0.800
Optimism 1995
Pessimism 1995
.322(.064)
-.261(.063)
9 1995 Maths Motivation
0.989
0.397
1995 Task Involvement
1995 Ego Orientation
0.89
.308(.070)
-.156(.060)
-.210(.062)
.180(.067)
Sex
1.00
1 Gender
-.148(.081)
Grade 1993
1.00
2 Student Grade 1993
.334(.074)
.828(.040)
-.150(.066)
10 Depression 1995
1.00
CDI 1995
0.91
-.117(.039)
School 1993
1.00
3 1993 School Attended
.098(.042)
.130(.060)
.132(.057)
11 1995 Maths Achievement
1.00
PAT Maths 1995
0.62
.681(.035)
-.208(.045)
0.78
6 1993 Maths Achievement
-.440(.071)
7 1994 Teacher Judgements
0.91
1.00
PATMaths 1993
0.905
0.927
Teacher Rating of Behaviour 1994
Achievement Rating 1994

Variance of criterion explained $R^2 = 0.62$ ($Q^2 = 0.60$)

Figure 2 Significant paths in the PLSPATH model

Results

In the path model shown in Figure 2, students' optimistic or pessimistic explanatory style in 1993 was moderately predictive of their explanatory style in 1995. Achievement in mathematics was weakly influenced by explanatory style through students' motivation towards mathematics in both years, with the relationship in 1995 also mediated by depression. Students' self-reported depression was weakly influenced by teachers' ratings in the previous year. Teachers' ratings were predictive of subsequent student achievement in mathematics. Gender was causally related to explanatory style in 1993 and teachers' ratings in 1994, with the pattern of males being more pessimistic and receiving lower ratings from teachers carrying through indirectly from poorer motivation towards mathematics, through depression to lower achievement in mathematics in 1995. While these causal relationships were generally weak, it was nevertheless clear that at least for some students, a more pessimistic outlook in primary school put them on a trajectory of poorer attitudes towards and lower achievement in mathematics.

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The grade level of the students' in 1993 was causally related to their explanatory style, achievement in mathematics and teachers' ratings, with older students being less optimistic. The 1993 primary school attended by the students had an impact on both the teachers' ratings and achievement in mathematics in 1995. Over the course of the study, approximately half all the students moved into secondary schools, but surprisingly this did not have an impact on their later achievement, once the effect of prior achievement had been taken into account. Overall the model explained approximately 62 per cent of the variance associated with students' achievement.

Summary of the Results

1 Students' explanatory style (pessimism) influences and is influenced by motivation.

2 Explanatory style exerts an indirect effect on students' achievement in mathematics through their motivation and through their depression.

3 Explanatory style is related to concurrent measures of depression, with this relationship enhanced by the addition of the students' motivation.

4 Teachers' ratings predict students' depression and subsequent achievement. Their ratings are influenced by students' prior achievement and motivation, as well as by the students' gender, grade level and primary school in 1993.

Discussion

Students' tendency to explain the causes of events from an optimistic or pessimistic framework influenced their achievement in mathematics; through their goal oriented motivation and self-reported depression. This is a significant finding, indicating that students' characteristic attributional patterns, established during the primary school years, had long term effects on their orientation to learning mathematics. Failure is integral to all learning, yet pessimistic students interpreted failure in mathematics as a negative experience likely to be long-lasting, pervasive and to be due to their ineptitudes. This outlook, which adversely influenced their motivational goals and academic behaviour observed by teachers in mathematics classrooms, was more evident in males. Pessimistic students were less willingness to engage in lessons and learning activities, had lower achievement and were more likely to report depression.

Implications for the Psychology of Mathematics Education

The finding that students' optimistic or pessimistic explanatory style influenced their motivation towards and achievement in mathematics is of immense significance for the psychology of mathematics education. Success and failure are highly salient in mathematics (McLeod, 1992), with mastery entailing hundreds of hours of sustained practice. Errors are an inevitable part of this learning, but students need to learn to view mistakes positively and maintain high levels of effortful responding.

Negative explanatory styles are learned but can be changed (Seligman, 1990). Students with a tendency to interpret failure in mathematics from a pessimistic perspective need to be identified as early as possible in their primary school years, so that their trajectory towards depression and poorer achievement can be interrupted and reversed. Teachers need to be cognisant of both students' attributions and those that they make about students' work, particularly in relation to failures.

A unique feature of this study is that motivation was measured directly in primary and lower secondary classrooms from the perspective of both teachers and students. The antecedents of student depression measured in the third year were evident in both students' ratings and in their disaffection with academic learning and lower achievement noted by teachers. Recent research evidence has related students' perception of failure at school to adolescent depression and suicide ideation (Martin, 1996). As rates of youth depression and suicide continue to be unacceptably high in many countries, understanding of the role of students' optimistic and pessimistic explanatory style in their motivation and achievement in mathematics is vital.

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